WXML Final Report: Algebraic Combinatorics

Faculty mentors: Sara Billey and Philippe Nadeau
Graduate mentor: Jordan Weaver
Undergraduates: Thomas Browning and Jesse Rivera

Spring 2018

1 Introduction

In this project we investigate a certain type of algebraic structure found in symmetric groups. Our work builds off of the theory developed in the paper "Parabolic Double Cosets in Coxeter Groups" by Sara Billey, Matjaž Konvalinka, T. Kyle Petersen, William Slofstra, and Bridget Tenner [1]. We will provide a brief summary of this theory in the following section for the purpose of defining terminology and recalling some useful results.

2 Background

We assume the reader is familiar with the symmetric group S_n. An *adjacent transposition* is a permutation that swaps a single pair of adjacent numbers. We write s_i to denote the adjacent transposition that swaps $(i \leftrightarrow i+1)$. For example, $s_3 = [12435] \in S_5$. A *parabolic subgroup* of S_n is one that is generated by adjacent transpositions. These are denoted by $W_I \coloneqq \langle s_i \mid i \in I \rangle$ for $I \subseteq \{1, \ldots, n-1\}$. A *parabolic double coset* is a two-sided coset with respect to two parabolic subgroups, $W_I w W_J \coloneqq \{pwq \mid p \in W_I, q \in W_J\}$.

For our purposes it is useful to think of symmetric groups as being generated by adjacent transpositions, $S_n = \langle s_i \mid i = 1, \ldots, n-1 \rangle$. In other words, any permutation can be written as a product of adjacent transpositions. If $w = s_{i_1} \cdots s_{i_k}$, we say that $s_{i_1} \cdots s_{i_k}$ is an *expression* for w. Moreover, if k is the smallest number of adjacent transpositions needed to write w, then we say that $s_{i_1} \cdots s_{i_k}$ is a *reduced expression* for w. This number, k, of adjacent transpositions in a reduced expression for w is defined to be the length of w, and we denote it by $\ell(w) = k$. In general, reduced expressions are *not* unique (not even up to reordering). For example, $s_3 s_4 s_3 = s_4 s_3 s_4$ are two distinct reduced expressions for $[12543] \in S_5$. We will make use of the following result from Corollary 1.4.8 in [2], which is reproduced below without proof.

Proposition 1. *Any expression $w = s_{i_1} \cdots s_{i_k}$ contains a reduced expression for w as a subword, obtainable by deleting an even number of letters.*

There happens to be a useful partial order that one can put on S_n. For two permutations $u, v \in S_n$, we write $u \leq v$ in *Bruhat order* if every reduced expression for v contains a subword that is a reduced expression for u. That is, if $v = s_{i_1} \cdots s_{i_k}$ is reduced, then there exists a reduced expression $u = s_{i_{a_1}} \cdots s_{i_{a_j}}$ for some $1 \leq a_1 \leq \cdots \leq a_j \leq k$. It turns out that every parabolic double coset is an interval in Bruhat order. That is, every parabolic double coset is of the form $[u, v] = \{w \in S_n : u \leq w \leq v\}$. This, in particular, implies that every parabolic double coset has a unique element of maximal length and a unique element of minimal length.

We define left and right ascent and descent sets of a permutation $w \in S_n$ as follows:

$$\begin{aligned}
\mathrm{Asc}_L(w) &= \{1 \leq i \leq n-1 : \ell(s_i w) > \ell(w)\} \\
\mathrm{Des}_L(w) &= \{1 \leq i \leq n-1 : \ell(s_i w) < \ell(w)\} \\
\mathrm{Asc}_R(w) &= \{1 \leq i \leq n-1 : \ell(w s_i) > \ell(w)\} \\
\mathrm{Des}_R(w) &= \{1 \leq i \leq n-1 : \ell(w s_i) < \ell(w)\}.
\end{aligned}$$

It can be shown that a permutation w is the minimal length element of the parabolic double coset $W_I w W_J$ if and only if $I \subseteq \mathrm{Asc}_L(w)$ and $J \subseteq \mathrm{Asc}_R(w)$. The following is another result from [2] (Corollary 1.4.6) that we will eventually use.

Proposition 2. *For all $i \in \{1, \ldots, n-1\}$ and $w \in S_n$, the following hold:*

i) $i \in \mathrm{Des}_L(w)$ if and only if some reduced expression for w begins with s_i.

ii) $i \in \mathrm{Des}_R(w)$ if and only if some reduced expression for w ends with s_i.

Lastly, we reproduce Corollary 2.11 in [1], which is a useful criteria for determining whether an interval is a parabolic double coset.

Proposition 3. *The interval $[u, v]$ is a parabolic double coset in S_n if and only if*

$$u = \min W_{\mathrm{Asc}_L(u) \cap \mathrm{Des}_L(v)} v W_{\mathrm{Asc}_R(u) \cap \mathrm{Des}_R(v)}.$$

The underlying question that motivates our work is simple: How many parabolic double cosets are in S_n? This question is answered in [1] in the (more general) context of finitely generated Coxeter groups. The presented approach involves computing the number of parabolic double cosets in S_n whose minimal element is w, and then summing this number over all permutations $w \in S_n$. One of the main reasons why this project exists is because the authors believe there to be a more efficient way of counting parabolic double cosets. Our primary goal is to find such a way. This quarter we explored the ideas of enumerating parabolic double cosets by their cardinality, rank (difference in lengths of maximal and minimal elements), and isomorphism type (in Bruhat order).

3 Intervals of Rank $\binom{n}{2} - 1$

Lemma 1. *Let G be a group with subgroups H and K, and let $g \in G$. If $x \in HgK$, then $HgK = HxK$.*

Proof. Since $x \in HgK$ we can write $x = hgk$ for some $h \in H$ and $k \in K$. Then $HgK = H(hxk)K = (Hh)x(kK) = HxK$. ☐

Lemma 2. *Let $n \geq 3$ and $w_0 \in S_n$ be the longest element. If $w_0 = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $1 < i < n-1$, then s_i appears in $s_{i_1} s_{i_2} \cdots s_{i_k}$ more than once. That is, every expression for w_0 contains more than one instance of s_i.*

Proof. Fix $1 < i < n-1$ and consider the permutation $u_i = s_i s_{i-1} s_{i+1} s_i \in S_n$. We find through brute force computation that $s_i s_{i-1} s_{i+1} s_i$ and $s_i s_{i+1} s_{i-1} s_i$ are the only reduced expressions for u_i. Notice that both expressions contain two instances of s_i. Then since $u_i \leq w_0$, every reduced expression for w_0 must contain two or more instances of s_i. Since every expression contains a reduced expression as a subword, this proves the claim. ☐

Lemma 3. *Let C be a subset of S_n. Then C is a parabolic double coset if and only if its image under the map $w \mapsto w_0 w$ is a parabolic double coset.*

Proof. Since the map $w \mapsto w_0 w$ is an involution we only need to prove one direction. Suppose $C \subseteq S_n$ is a parabolic double coset. Then $C = W_I w W_J$ for some $w \in S_n$ and $I, J \subseteq \{1, \ldots, n-1\}$. We will show that $w_0(W_I w W_J) = W_{w_0(I)} w_0 w W_J$. Indeed, if $x \in w_0(W_I w W_J)$ then we can write $x = w_0(s_{i_1} \cdots s_{i_a}) w (s_{j_1} \cdots s_{j_b})$ for some $i_1, \ldots, i_a \in I$ and $j_1, \ldots, j_b \in J$. Then

$$\begin{aligned} x &= w_0(s_{i_1} \cdots s_{i_a}) w (s_{j_1} \cdots s_{j_b}) \\ &= (s_{n-i_1} \cdots s_{n-i_a}) w_0 w (s_{j_1} \cdots s_{j_b}) \\ &= (s_{w_0(i_1)} \cdots s_{w_0(i_a)}) w_0 w (s_{j_1} \cdots s_{j_b}) \end{aligned}$$

so $x \in W_{w_0(I)} w_0 w W_J$. Conversely, if $y \in W_{w_0(I)} w_0 w W_J$ then $y = (s_{w_0(k_1)} \cdots s_{w_0(k_c)}) w_0 w (s_{l_1} \cdots s_{l_d})$ for some $c, d \in \mathbb{N}$, $k_1, \ldots, k_c \in I$, and $l_1, \ldots, l_d \in J$. Then

$$\begin{aligned} y &= (s_{w_0(k_1)} \cdots s_{w_0(k_c)}) w_0 w (s_{l_1} \cdots s_{l_d}) \\ &= w_0 (s_{n-w_0(k_1)} \cdots s_{n-w_0(k_c)}) w (s_{l_1} \cdots s_{l_d}) \\ &= w_0 (s_{k_1} \cdots s_{k_c}) w (s_{l_1} \cdots s_{l_d}) \end{aligned}$$

and therefore $y \in w_0(W_I w W_J)$. We have shown $W_{w_0(I)} w_0 w W_J \subseteq w_0(W_I w W_J)$ and $w_0(W_I w W_J) \subseteq W_{w_0(I)} w_0 w W_J$, hence the two sets are equal. ☐

Theorem 1. *For $n \geq 3$, there are exactly 4 parabolic double cosets in S_n of rank $\binom{n}{2} - 1$.*

Proof. Let $e \in S_n$ denote the identity. First note that any interval in S_n of rank $\binom{n}{2} - 1$ is of the form $[s_i, w_0]$ or $[e, w_0 s_i]$ for some adjacent transposition s_i, $i \in \{1, \ldots, n-1\}$. We will show that $i = 1, n-1$ are the only choices of i that correspond to parabolic double cosets. To see that the intervals $[e, w_0 s_1], [e, w_0 s_{n-1}], [s_1, w_0]$, and $[s_{n-1}, w_0]$ are indeed parabolic double cosets, observe that

$$\begin{aligned}
[s_1, w_0] &= W_{\{2,\ldots,n-1\}} w_0 W_{\{2,\ldots,n-1\}} \\
[s_{n-1}, w_0] &= W_{\{1,\ldots,n-2\}} w_0 W_{\{1,\ldots,n-2\}} \\
[e, w_0 s_1] &= W_{\{1,\ldots,n-2\}} e W_{\{2,\ldots,n-1\}} \\
[e, w_0 s_{n-1}] &= W_{\{2,\ldots,n-1\}} e W_{\{1,\ldots,n-2\}}.
\end{aligned}$$

Now fix $i \in \{2, \ldots, n-2\}$ and consider the interval $[s_i, w_0]$. Suppose for the sake of contradiction that this interval is a parabolic double coset. Then there exist $w \in S_n$ and $I, J \subseteq \{1, \ldots, n-1\}$ such that $s_i = \min W_I w W_J$ and $w_0 = \max W_I w W_J$. In particular, $s_i, w_0 \in W_I w W_J$ and $I \subseteq \mathrm{Asc}_L(s_i)$ and $J \subseteq \mathrm{Asc}_R(s_i)$. By Lemma 1 we can write $W_I w W_J = W_I s_i W_J$. Then since $w_0 \in W_I s_i W_J$ we have $w_0 = (s_{i_1} \cdots s_{i_a}) s_i (s_{j_1} \cdots s_{j_b})$ for some $a, b \in \mathbb{N}$, $i_1, \ldots, i_a \in I$ and $j_1, \ldots, j_b \in J$. Then Lemma 2 tells us that $i \in \{i_1, \ldots, i_a, j_1, \ldots, j_b\}$. But since $s_i^2 = e$ and $\ell(e) < \ell(s_i)$, we know that $i \notin \mathrm{Asc}_L(s_i) \cup \mathrm{Asc}_R(s_i)$ and consequently $i \notin I \cup J$. This gives us a contradiction.

That $[e, w_0 s_i]$ is not a parabolic double coset follows from Lemma 3 and the fact that the map $w \mapsto w_0 w$ is an antiautomorphism of Bruhat order that sends $[e, w_0 s_i] \mapsto [s_i, w_0]$. ☐

4 Intervals of Rank $\binom{n}{2} - 2$

Lemma 4. *For $n \geq 5$, the intervals $[s_1 s_{n-1}, w_0]$ and $[e, w_0 s_1 s_{n-1}]$ are not parabolic double cosets in S_n.*

Proof. By Lemma 3 it is sufficient to show that $[s_1 s_{n-1}, w_0]$ is not a parabolic double coset. Since $n \geq 5$, $|1 - (n-1)| > 1$ so that s_1 and s_{n-1} commute and $\mathrm{Asc}_L(s_1 s_{n-1}) \cap \mathrm{Des}_L(w_0) = \mathrm{Asc}_R(s_1 s_{n-1}) \cap \mathrm{Des}_R(w_0) = \{2, \ldots, n-2\}$. It is clear that $\min W_{\{2,\ldots,n-2\}} w_0 W_{\{2,\ldots,n-2\}} = t_{1,n} = [n, 2, 3, \ldots, n-2, n-1, 1] \neq s_1 s_{n-1}$, so by Proposition 3, this implies that $[s_1 s_{n-1}, w_0]$ is not a parabolic double coset. ☐

Lemma 5. *Let $n \geq 5$. If $|i - j| > 1$ then $[s_i s_j, w_0]$ and $[e, w_0 s_i s_j]$ are not parabolic double cosets in S_n.*

Proof. By Lemma 3 we only need to check $[s_i s_j, w_0]$. First note that s_i and s_j commute so that $\mathrm{Des}_L(s_i s_j) = \mathrm{Des}_R(s_i s_j) = \{i, j\}$. If both $i, j \in \{1, n-1\}$ then the result follows from the previous lemma, so we may assume this is not the case. Suppose for contradiction that $[s_i s_j, w_0]$ is a parabolic double coset. Then $w_0 = (s_{i_1} \cdots s_{i_a}) s_i s_j (s_{j_1} \cdots s_{j_b})$ for some $a, b \in \mathbb{N}$, $i_1, \ldots, i_a \in \mathrm{Asc}_L(s_i s_j)$, and $j_1, \ldots, j_b \in \mathrm{Asc}_R(s_i s_j)$. Then since $|i - j| > 1$ and it is not the case that both $i, j \in \{1, n-1\}$, we have that either $1 < i < n-1$ or $1 < j < n-1$. Suppose without loss of generality that $1 < i < n-1$. By Lemma 2, we know that every expression for w_0 contains more than one instance of s_i, so $i \in \{i_1, \ldots, i_a, j_1, \ldots, j_b\}$. But this gives us a contradiction, since $i \in \mathrm{Des}_L(s_i s_j)$ while $\{i_1, \ldots, i_a, j_1, \ldots, j_b\} \subseteq \mathrm{Asc}_L(s_i s_j) = \mathrm{Asc}_R(s_i s_j)$. ☐

Lemma 6. *Let $x, y \in S_n$, $I, J \subseteq \{1, \ldots, n-1\}$, $m_x = \min W_I x W_J$, and $m_y = \min W_I y W_J$. If $x \leq y$, then $\ell(m_x) \leq \ell(m_y)$.*

Proof. Consider the following procedure, which is a slight modification of the greedy algorithm outlined in Corollary 2.10 of [1] for finding the minimal element of a parabolic double coset. If $I \cap \mathrm{Des}_L(y) \neq \emptyset$, then there exists a reduced expression $y = s_d \cdot s_{i_1} \cdots s_{i_a}$ where $d \in I \cap \mathrm{Des}_L(y)$ (this follows from Proposition 2). Since $x \leq y$, this reduced expression for y contains a reduced expression for x as a subword. If this reduced expression for x begins with s_d, we multiply both x and y on the left by s_d. Otherwise, we only multiply y on the left by s_d. Let x' and y' denote the resulting permutations. It is clear that in both cases we still have $x' \leq y'$. If $J \cap \mathrm{Des}_R(y) \neq \emptyset$ we do the same but on the right. Call the resulting permutations x'' and y'', and again notice that we still have $x'' \leq y''$. We continue this process until the algorithm terminates at m_y (we know from Corollary 2.10 in [1] that this algorithm does in fact terminate, and that it ends at m_y). This leaves us with a permutation $v = x^{''\cdots'} \in W_I x W_J$ such that $v \leq m_y$. It follows that $\ell(m_x) \leq \ell(v) \leq \ell(m_y)$ (since m_x is the minimal element in $W_I x W_J$ and $v \in W_I x W_J$). ☐

Lemma 7. *Let $n \geq 5$ and $i \neq j$. If $2 < i < n-2$ or $2 < j < n-2$ then $[s_i s_j, w_0]$ and $[e, w_0 s_i s_j]$ are not parabolic double cosets in S_n.*

Proof. By Lemma 3 we only need to check $[s_i s_j, w_0]$. The case in which $|i-j| > 1$ is covered in Lemma 5, so we may assume that either $j = i+1$ or $i = j+1$.

Suppose first that $j = i+1$. Consider the permutation $v_i = s_i s_{i+1} s_{i-1} s_i s_{i+2} s_{i+1} \in S_n$. One can verify through computation that $\mathrm{Des}_L(v_i) = \{i\}$ and $\mathrm{Des}_R(v_i) = \{i+1\}$. It follows that if $I = \{1, \ldots, n-1\} \setminus \{i\}$ and $J = \{1, \ldots, n-1\} \setminus \{i+1\}$ then $\min W_I v_i W_J = v_i$. Let $m = \min W_I w_0 W_J$. Then by Lemma 6, it must be the case that $\ell(m) \geq \ell(v_i) = 6$. This, in particular, implies that $m \neq s_i s_{i+1}$. Since $I = \mathrm{Asc}_L(s_i s_{i+1}) \cap \mathrm{Des}_L(w_0)$ and $J = \mathrm{Asc}_R(s_i s_{i+1}) \cap \mathrm{Des}_R(w_0)$, it follows from Proposition 3 that $[s_i s_{i+1}, w_0]$ is not a parabolic double coset.

If $i = j+1$ we instead consider $v_j^{-1} = s_{j+1} s_{j+2} s_j s_{j-1} s_{j+1} s_j \in S_n$ and proceed as in the previous case to reach the same conclusion. □

Lemma 8. *Let $n \geq 6$ and $w_0 \in S_n$ be the longest element. If $w_0 = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $2 < i < n-2$, then s_i appears in $s_{i_1} s_{i_2} \cdots s_{i_k}$ more than twice. That is, every expression for w_0 contains more than two instance of s_i.*

Proof. Consider the permutation $z_i = s_i s_{i-1} s_{i+1} s_i s_{i+2} s_{i+1} s_{i-2} s_{i-1} s_i \in S_n$. We find through computation that all 42 reduced expressions for z_i contain 3 instances of s_i. Then since $z_i \leq w_0$, we know that all reduced expressions for w_0 contain at least 3 instances of s_i. Since all expressions contain a reduced expression as a subword, this proves the claim. □

Lemma 9. *Let $n \geq 5$. If $1 < i < n-1$ or $1 < j < n-1$ then $[s_i, w_0 s_j]$ is not a parabolic double coset in S_n.*

Proof. Suppose for contradiction that $1 < i < n-1$ and $[s_i, w_0 s_j]$ is a parabolic double coset. Note that $\mathrm{Des}_L(s_i) = \mathrm{Des}_R(s_i) = \{i\}$. Then $(s_{i_1} \cdots s_{i_a}) s_i (s_{j_1} \cdots s_{j_b}) = w_0 s_j$ for some $a, b \in \mathbb{N}$, $i_1, \ldots, i_a \in \mathrm{Asc}_L(s_i)$, and $j_1, \ldots, j_b \in \mathrm{Asc}_R(s_i)$. Multiplying on the right by s_j this becomes $(s_{i_1} \cdots s_{i_a}) s_i (s_{j_1} \cdots s_{j_b}) s_j = w_0$. But since $i \in \mathrm{Des}_L(s_i) \cap \mathrm{Des}_R(s_i)$ we know that $i \notin \{i_1, \ldots i_a, j_1, \ldots, j_b\}$. If $i \neq j$, this contradicts the fact that every expression for w_0 contains more than one instance of s_i (Lemma 2). If $i = j \neq \frac{n}{2}$ then we can use the fact that $w_0 s_i = s_{n-i} w_0$ to obtain $s_{n-i}(s_{i_1} \cdots s_{i_a}) s_i (s_{j_1} \cdots s_{j_b}) = w_0$, which again contradicts Lemma 2 since $n - i \neq i$. If $i = j = \frac{n}{2}$, then n is even and we cannot use this approach (since w_0 and $s_{n/2}$ commute). We will show that in this case there is a contradiction with lemma 8. Since $n \geq 6$ (due to the fact that n is always even in this case), we have that $2 < n/2 < n-2$. It then follows from Lemma 8 that every expression for w_0 contains at least 3 instances of $s_{n/2}$. But we still have that $s_{n/2}(s_{i_1} \cdots s_{i_a}) s_{n/2} (s_{j_1} \cdots s_{j_b}) = w_0$, which gives us our desired contradiction.

If $i \in \{1, n-1\}$ and $1 < j < n-1$ then we use the map $w \mapsto w_0 w$ to end up in the previous case. □

Theorem 2. *For $n \geq 5$, there are exactly 12 parabolic double cosets in S_n of rank $\binom{n}{2} - 2$.*

Proof. It is a straightforward computation to verify that

$$
\begin{aligned}
&[s_1 s_2, w_0] = W_{\{2,\ldots,n-1\}} w_0 W_{\{1,3,\ldots,n-1\}} && (1)\\
&[s_2 s_1, w_0] = W_{\{1,3,\ldots,n-1\}} w_0 W_{\{2,\ldots,n-1\}} && (2)\\
&[s_{n-2} s_{n-1}, w_0] = W_{\{1,\ldots,n-3,n-1\}} w_0 W_{\{1,\ldots,n-2\}} && (3)\\
&[s_{n-1} s_{n-2}, w_0] = W_{\{1,\ldots,n-2\}} w_0 W_{\{1,\ldots,n-3,n-1\}} && (4)\\
&[e, w_0 s_2 s_1] = W_{\{1,\ldots,n-3,n-1\}} e W_{\{2,\ldots,n-1\}} && (5)\\
&[e, w_0 s_1 s_2] = W_{\{1,\ldots,n-2\}} e W_{\{1,3,\ldots,n-1\}} && (6)\\
&[e, s_1 s_2 w_0] = W_{\{2,\ldots,n-1\}} e W_{\{1,\ldots,n-3,n-1\}} && (7)\\
&[e, s_2 s_1 w_0] = W_{\{1,3,\ldots,n-1\}} e W_{\{1,\ldots,n-2\}} && (8)\\
&[s_1, w_0 s_1] = W_{\{2,\ldots,n-2\}} s_1 W_{\{2,\ldots,n-1\}} && (9)\\
&[s_1, w_0 s_{n-1}] = W_{\{2,\ldots,n-1\}} s_1 W_{\{2,\ldots,n-2\}} && (10)\\
&[s_{n-1}, w_0 s_1] = W_{\{1,\ldots,n-2\}} s_{n-1} W_{\{2,\ldots,n-2\}} && (11)\\
&[s_{n-1}, w_0 s_{n-1}] = W_{\{2,\ldots,n-2\}} s_{n-1} W_{\{1,\ldots,n-2\}}. && (12)
\end{aligned}
$$

The previous 6 lemmas show that these are indeed the only parabolic double cosets of rank $\binom{n}{2} - 2$. This can be seen by first noting that all intervals of rank $\binom{n}{2} - 2$ are of the form

$$[s_i s_j, w_0] \qquad (i \neq j) \tag{13}$$
$$[e, w_0 s_i s_j] \qquad (i \neq j) \tag{14}$$
$$[s_i, w_0 s_j]. \tag{15}$$

Note that intervals of type (13) and type (14) correspond under the map $w \mapsto w_0 w$, so by Lemma 3 it is sufficient to only consider intervals of types (13) and (15). In Lemma 9 we showed that intervals of type (15) can only be parabolic double cosets if both $i, j \in \{1, n-1\}$. It turns out that all four choices of (i, j) are indeed parabolic double cosets, as seen above in (9) - (12).

Next, we want to show that parabolic double cosets of type (13) necessarily have either both $i, j \in \{1, 2\}$ or $i, j \in \{n-1, n-2\}$. Lemma 5 shows that we cannot have $|i - j| > 1$, and Lemma 7 shows that that both $i, j \in \{1, 2, n-1, n-2\}$. Together with the assumption that $n \geq 5$, these give the desired result. It is shown above in (1) - (4) that all intervals of type (13) satisfying this condition are indeed parabolic double cosets. Lastly, (5) - (8) are just the images of (1) - (4) under the map $w \mapsto w_0 w$. □

The following results prove a slightly more general version of Lemma 7.

Lemma 10. *Let $w \in S_n$ and $I, J \subseteq \{1, \ldots, n-1\}$. If $I' \subseteq I$ and $J' \subseteq J$ then $\ell(\min W_I w W_J) \leq \ell(\min W_{I'} w W_{J'})$.*

Proof. This follows immediately from the fact that $W_{I'} w W_{J'} \subseteq W_I w W_J$ by definition. □

Lemma 11. *Let $n \geq 5$, $1 < i < n-2$, and $I, J \subseteq \{1, \ldots, n-1\}$. If $i \notin I$ and $i+1 \notin J$ then $\ell(\min W_I w_0 W_J) \geq 6$.*

Proof. We saw in the proof of Lemma 7 that $\ell(\min W_{\{1,\ldots,n-1\} \setminus \{i\}} w_0 W_{\{1,\ldots,n-1\} \setminus \{i+1\}}) \geq 6$, so if $I \subseteq \{1, \ldots, n-1\} \setminus \{i\}$ and $J \subseteq \{1, \ldots, n-1\} \setminus \{i+1\}$ then we have by Lemma 10 that

$$\ell(\min W_I w_0 W_J) \geq \ell(\min W_{\{1,\ldots,n-1\} \setminus \{i\}} w_0 W_{\{1,\ldots,n-1\} \setminus \{i+1\}}) \geq 6.$$

□

Lemma 12. *Let $n \geq 5$ and $1 < i < n-2$. If $u, v \in S_n$ such that $i \in \mathrm{Des}_L(u)$, $i+1 \in \mathrm{Des}_R(u)$, $\ell(u) \leq 5$, and $v \geq s_i s_{i+1} s_{i-1} s_i s_{i+2} s_{i+1}$, then $[u, v]$ is not a parabolic double coset in S_n.*

Proof. Let $u, v \in S_n$ be such permutations. Then

$$\ell(\min W_{\mathrm{Asc}_L(u) \cap \mathrm{Des}_L(v)} v W_{\mathrm{Asc}_R(u) \cap \mathrm{Des}_R(v)}) \geq \ell(\min W_{\mathrm{Asc}_L(u)} w_0 W_{\mathrm{Asc}_R(u)}) \geq 6$$

by Lemmas 6, 10, and 11. Since $\ell(u) \leq 5$ by assumption, this means $u \neq \min W_{\mathrm{Asc}_L(u) \cap \mathrm{Des}_L(v)} v W_{\mathrm{Asc}_R(u) \cap \mathrm{Des}_R(v)}$ and consequently $[u, v]$ is not a parabolic double coset by Proposition 3. □

5 Parabolic Representations

Let $C = W_I w W_J$ be a parabolic double coset in S_n. Then $C w^{-1} = W_I (w W_J w^{-1})$ can be viewed as a collection of permutations of the values $\{1, \ldots, n\}$. Let V denote the subset of the values $\{1, \ldots, n\}$ that are not fixed by $C w^{-1}$. Let H_L denote the subgroup of $\mathfrak{S}_V$ given by restricting W_I to V and let H_R denote the subgroup of $\mathfrak{S}_V$ given by restricting $w W_j w^{-1}$ to V. Let $A_L \subseteq V \times V$ denote the collection of pairs (v_1, v_2) of values such that the values v_1 and v_2 are adjacent in $\{1, \ldots, n\}$. Let $A_R \subseteq V \times V$ denote the collection of pairs (v_1, v_2) of values such that the positions $w^{-1} v_1$ and $w^{-1} v_2$ are adjacent in $\{1, \ldots, n\}$. The set I corresponds to the subset $T_L \subseteq A_L$ of pairs (v_1, v_2) of values such that the transposition swapping the values v_1 and v_2 lies in I. The set J corresponds to the subset $T_R \subseteq A_R$ of pairs (v_1, v_2) of values such that the transposition swapping the positions $w^{-1} v_1$ and $w^{-1} v_2$ lies in J. The tuple $\Phi = (V, A_L, A_R, T_L, T_R)$ is an example of a parabolic representation that encodes C at w.

Definition 1. A parabolic representation consists of a finite collection V of letters, left adjacency relations $A_L \subseteq V \times V$, right adjacency relations $A_R \subseteq V \times V$, left transpositions $T_L \subseteq A_L$, and right transpositions $T_R \subseteq A_R$ such that the graphs (V, A_L) and (V, A_R) are linear forests (disjoint unions of paths) and such that every element of V is contained in some element of $T_L \cup T_R$.

Definition 2. If $\Phi = (V, A_L, A_R, T_L, T_R)$ is a parabolic representation, H_L the group of permutations of V generated by the left transpositions T_L, and H_R the group of permutations of V generated by the right transpositions T_R, then we define $\Pi_\Phi = H_L H_R$.

Lemma 13. *If* $\Phi = (V, A_L, A_R, T_L, T_R)$ *and* $\Psi = (V, A_L, A_R, U_L, U_R)$ *are parabolic representations with* $\Pi_\Phi = \Pi_\Psi$ *then* $\Phi \cup \Psi = (V, A_L, A_R, T_L \cup U_L, T_R \cup U_R)$ *is a parabolic representation with* $\Pi_{\Phi\cup\Psi} = \Pi_\Phi = \Pi_\Psi$*.*

Proof. It is clear from the definitions that $\Phi \cup \Psi$ is a parabolic representation with $\Pi_\Phi = \Pi_\Psi \subseteq \Pi_{\Phi\cup\Psi}$. Note that Π_Φ is closed under left-multiplication by transpositions in T_L and Π_Ψ is closed under left-multiplication by transpositions in U_L so $\Pi_\Phi = \Pi_\Psi$ is closed under left-multiplication by transpositions in $T_L \cup U_L$. Similarly, $\Pi_\Phi = \Pi_\Psi$ is closed under right-multiplication by transpositions in $T_R \cup U_R$. However, $\Pi_{\Phi\cup\Psi}$ is the smallest collection of permutations of V containing the identity and closed under left-multiplication by $T_L \cup U_L$ and closed under right-multiplication by $T_R \cup U_R$. This shows that $\Pi_{\Phi\cup\Psi} \subseteq \Pi_\Phi = \Pi_\Psi$. □

Definition 3. If $\Phi = (V, A_L, A_R, T_L, T_R)$ is a parabolic representation then we define $\overline{\Phi}$ to be the maximal parabolic representation $\Psi = (V, A_L, A_R, U_L, U_R)$ with $\Pi_\Phi = \Pi_\Psi$. We say that Φ is maximal when $\Phi = \overline{\Phi}$.

Definition 4. If $\Phi = (V, A_L, A_R, T_L, T_R)$ and $\Psi = (W, B_L, B_R, U_L, U_R)$ are parabolic representations then an isomorphism between Φ and Ψ is a bijective function $V \to W$ such that the product map $V \times V \to W \times W$ takes A_L to B_L, A_R to B_R, T_L to U_L, and T_R to U_R.

If $\varphi\colon V \to \{1, \ldots, n\}$ is injective then we can define the injective homomorphism $\widetilde{\varphi}\colon \mathfrak{S}_V \to S_n$ by

$$\widetilde{\varphi}(\pi)(k) = \begin{cases} (\varphi \circ \pi \circ \varphi^{-1})(k) & k \in \varphi[V] \\ k & k \notin \varphi[V] \end{cases}.$$

Definition 5. If $\Phi = (V, A_L, A_R, T_L, T_R)$ is a parabolic representation and if C is a parabolic double coset of S_n and if w is an element of C then an encoding of C at w by Φ consists of injective functions $\varphi_L\colon V \to \{1, \ldots, n\}$ and $\varphi_R\colon V \to \{1, \ldots, n\}$ such that

1. $\varphi_L(u)$ is adjacent to $\varphi_L(v)$ if and only if $(u, v) \in A_L$ for all $u, v \in V$.
2. $\varphi_R(u)$ is adjacent to $\varphi_R(v)$ if and only if $(u, v) \in A_R$ for all $u, v \in V$.
3. $w(\varphi_R(v)) = \varphi_L(v)$ for all $v \in V$.
4. $\widetilde{\varphi_L}[\Pi_\Phi] = Cw^{-1}$.

Definition 6. Let C be a parabolic double coset of S_n and let w be an element of C. An isomorphism between encodings $(\Phi, \varphi_L, \varphi_R)$ and (Ψ, ψ_L, ψ_R) of C at w consists of an isomorphism f between Φ and Ψ such that $\psi_L \circ f = \varphi_L$ and $\psi_R \circ f = \varphi_R$.

Lemma 14. *Let* C *be a parabolic double coset of* S_n *and let* w *be an element of* C*. Let* V *denote the collection of values in* $\{1, \ldots, n\}$ *acted on by* Cw^{-1}*. Let* $A_L \subseteq V \times V$ *denote the collection of pairs* (v_1, v_2) *of values such that the values* v_1 *and* v_2 *are adjacent in* $\{1, \ldots, n\}$*. Let* $A_R \subseteq V \times V$ *denote the collection of pairs* (v_1, v_2) *of values such that the positions* $w^{-1}v_1$ *and* $w^{-1}v_2$ *are adjacent in* $\{1, \ldots, n\}$*. Let* $\varphi_L\colon V \to \{1, \ldots, n\}$ *be the inclusion map and let* $\varphi_R\colon V \to \{1, \ldots, n\}$ *be given by* $\varphi_R(v) = g^{-1}(v)$*.*
Then every encoding of C *at* w *is isomorphic to a unique encoding of* C *at* w *of the form*

$$((\Phi, A_L, A_R, T_L, T_R), \varphi_L, \varphi_R)$$

for some $T_L \subseteq A_L$ *and* $T_R \subseteq A_R$*. Call a choice of* (T_L, T_R) *valid if* $((V, A_L, A_R, T_L, T_R), \varphi_L, \varphi_R)$ *is an encoding of* C *at* w*. We may partially order valid choices of* (T_L, T_R) *by inclusion. Call a choice of* (I, J) *valid if* $C = W_I w W_J$*. We may partially order valid choices of* (I, J) *by inclusion. Then* φ_L *induces a poset isomorphism between valid choices of* (T_L, T_R) *and valid choices of* (I, J)*.*
In particular, there is a unique maximal parabolic representation that encodes C *at* w*, up to isomorphism.*

Proof. Let $((W, B_L, B_R, U_L, U_R), \psi_L, \psi_R)$ be an encoding of C at w. If $f\colon W \to V$ induces an isomorphism between $((W, B_L, B_R, U_L, U_R), \psi_L, \psi_R)$ and $((V, A_L, A_R, T_L, T_R), \phi_L, \phi_R)$ then $\phi_L \circ f = \psi_L$ so f is given by ψ_L. This shows the uniqueness part of the first statement. For the existence part of the first statement, the bijection $\psi_L\colon W \to \psi_L[W]$ induces an isomorphism on $((W, B_L, B_R, U_L, U_R), \psi_L, \psi_R)$. Then to prove the existence part of the first statement, we may assume without loss of generality that $W \subseteq \{1, \ldots, n\}$ and that ψ_L is the inclusion map. We now show that the definitions of a parabolic representation will force $W = V$, $B_L = A_L$, $B_R = A_R$, $\psi_L = \varphi_L$, and $\psi_R = \varphi_R$. First note that W is the collection of letters acted on by Π_Φ. By condition 4 of definition 5, this is also the collection of letters acted on by Cw^{-1}. This shows that $W = V$. Also, condition 2 of definition 5 states that $w(\psi_R(v)) = v$ for all $v \in V$ or, equivalently, that $\psi_R(v) = w^{-1}(v)$ for all $v \in V$. This shows that $\psi_R = \varphi_R$. Then conditions 1 and 2 of definition 5 become

1. u is adjacent to v if and only if $(u, v) \in B_L$ for all $u, v \in V$.
2. $w^{-1}(u)$ is adjacent to $w^{-1}(v)$ if and only if $(u, v) \in B_R$ for all $u, v \in V$.

Then $B_L = A_L$ and $B_R = A_R$ which completes the proof of the existence part of the first statement.
Note that a choice of (T_L, T_R) is valid if and only if $\widetilde{\varphi_L}[H_L H_R] = Cw^{-1}$ and note that a choice of (I, J) is valid if and only if $C = W_I w W_J$ if and only if $W_I(wW_J w^{-1}) = Cw^{-1}$. However, if φ_L takes (T_L, T_R) to (I, J) then $\widetilde{\varphi_L}[H_L] = W_I$ and $\widetilde{\varphi_L}[H_R] = wW_J w^{-1}$. This shows the second statement. Lemma 13 and definition 3 show that the poset of valid choices of (T_L, T_R) has a maximal element. As a consequence, the poset of valid choices of (I, J) also has a maximal element (the maximal presentation). ☐

Lemma 15. *Let $\Phi = (V, A_L, A_R, T_L, T_R)$ be a parabolic representation. Let $c_\Phi(n)$ count the number of pairs (C, w) consisting of a parabolic double coset C in S_n and an element w of C such that Φ encodes C at w. Then $c_\Phi(n)$ is given by*

$$c_\Phi(n) = \frac{1}{\mathrm{Aut}(V, A_L, A_R, T_L, T_R)} 2^{p_L + p_R - s_L - s_R} (n - |V|)! \frac{(n - |V| + 1)!}{(n - |V| - p_L + 1)!} \frac{(n - |V| + 1)!}{(n - |V| - p_R + 1)!}$$

where p_j denotes the number of connected components of A_j for $j \in \{L, R\}$ and where s_R denotes the number of isolated vertices of A_j for $j \in \{L, R\}$. Alternatively, if A_j is thought of as a inducing a partition of S for $j \in \{L, R\}$ then p_j denotes the number of parts of A_j for $j \in \{L, R\}$ and s_j denotes the number of parts of cardinality 1 for $j \in \{L, R\}$.

Proof. Let $j \in \{L, R\}$ and let A_j have connected components of orders $a_1, a_2, \ldots, a_{p_j}$. The ways to embed distinguishable chains of lengths $a_1, a_2, \ldots, a_{p_j}$ into $\{1, \ldots, n\}$ such that distinct chains are not adjacent are in bijection with ways to place p_j distinguished balls into $n - |V| + 1$ boxes. Then the number of embeddings $\varphi_j\colon S \to \{1, \ldots, n\}$ that preserve adjacency is given by $2^{p_j - s_j}(n - |V| + 1)!/(n - |V| - p_j + 1)!$ where $p_j - s_j$ is the number of connected componenets of A_j of cardinality at least 2 where the orientation of the embedding matters. We multiply this quantity for $j = L$ and $j = R$ and then divide by $\mathrm{Aut}(V, A_L, A_R, T_L, T_R)$ as applying an automorphism of (V, A_L, A_R, T_L, T_R) would give an identical embedding. Conversely, if some embedding is overcounted then lemma 14 gives that the overcounting is given by applying an automorphism of (V, A_L, A_R, T_L, T_R). Finally, the number of w which satisfy condition 3 of definition 5 is given by $(n - |V|)!$ since we specify the values of w at $|V|$ positions. ☐

Theorem 3. *Let k be a natural number. Let $c_k(n)$ count the number of parabolic double cosets in S_n of cardinality k. Then $c_k(n)$ is given by a polynomial times a factorial.*

Proof. Note that $kc_k(n)$ counts the number of pairs (C, w) consisting of a parabolic double coset of S_n of cardinality k and an element w of C. Lemma 14 gives that any such pair is encoded by a unique maximal parabolic representation up to isomorphism. Then $kc_k(n)$ is given by summing $c_\Phi(n)$ over all isomorphism classes of maximal parabolic representations Φ with $|\Pi_\Phi| = k$. Lemma 15 gives that this sum is a polynomial times a factorial. ☐

6 Applications

Lemma 16. *Let W_I be a parabolic subgroup of S_n. If p is a prime and if $p^k \mid |W|$ then $p!^k \mid |W|$.*

Proof. Note that $|W|$ is a product of factorials so it suffices to show that if $p^k \mid m!$ then $p!^k \mid m!$. By de Polignac's formula for the power of a prime dividing a factorial, we may take

$$k = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \ldots.$$

However, there is a subgroup of S_m of order $p!^k$. First group some of the m letters into $\lfloor m/p \rfloor$ blocks of size p and allow for arbitrary permutations within each block. The resulting group has order $p^{\lfloor m/p \rfloor}$. Then group some of the $\lfloor m/p \rfloor$ blocks of size p into $\lfloor m/p^2 \rfloor$ "mega-blocks" (consisting of p blocks each) and allow for arbitrary permutations of the p blocks within each "mega-block". The resulting group has order $p^{\lfloor m/p \rfloor + \lfloor m/p^2 \rfloor}$. Repeating this construction gives the desired subgroup. □

Lemma 17. *Let $C = W_I w W_J$ be a parabolic double coset in S_n. Then*

$$|W_I| \mid |C|, \qquad |W_J| \mid |C|, \qquad |C| \mid |W_I||W_J|.$$

If p is a prime and if $p \mid |C|$ then $p! \mid |C|$. More generally, if $p^{2k-1} \mid |C|$ then $p!^k \mid |C|$.

Proof. For the first statement, note that C is a disjoint union of right cosets of C. For the second statement, note that C is a disjoint union of left cosets of C. For the third statement, note that $W_I \times W_J$ acts transitively on C and apply the orbit-stabilizer theorem. Then the final statement follows from the previous lemma. □

Lemma 18. *Let $\Phi = (V, A_L, A_R, T_L, T_R)$ be a maximal parabolic representation with $|V| = n$. Then*

$$2^{\lceil n/2 \rceil} \leq |\Pi_\Phi| \leq n!.$$

Moreover, if $|\Pi_\Phi| = n!$ then $\Pi_\Phi = \mathfrak{S}_V$ and $T_L = A_R$ and $T_R = A_R$.

Proof. Note that $\Pi_\Phi \subseteq \mathfrak{S}_V$ so $|\Pi_\Phi| \leq |\mathfrak{S}_V| = n!$. If $|\Pi_\Phi| = n!$ then $\Pi_\Phi = \mathfrak{S}$. Moreover, if $|\Pi_\Phi| = n!$ then $\Psi = (V, A_L, A_R, A_L, A_R)$ is a parabolic repereserentation with $\mathfrak{S}_V = \Pi_\Phi \subseteq \Pi_\Psi \subseteq \mathfrak{S}_V$ so $\Pi_\Phi = \Pi_\Psi$. Then the maximality of Φ gives that $\Phi = \Psi$ so $T_L = A_L$ and $T_R = A_R$.

For the other inequality, we may assume that Φ is chosen such that $|\Pi_\Phi|$ is minimal and then that the number of edges of T_L plus the number of edges of T_R is minimal. If $T_L \cup T_R$ contains a path or cycle of three edges then removing the middle edge (or any edge, in the case of a cycle) gives a contradiction. Then the graphs (V, T_L) and (V, T_R) are disjoint unions of paths of at most two edges. Suppose that $v_1 \sim v_2 \sim v_3$ is a path of two edges in (V, T_L). If v_2 is not isolated in (V, T_R) then replacing $v_1 \sim v_2 \sim v_3$ by $v_1 \sim v_3$ in (V, T_L) gives a contradiction. Otherwise, all of v_1, v_2, and v_3 are isolated in (V, T_R) so replacing $v_1 \sim v_2 \sim v_3$ by $v_1 \sim v_2$ in (V, T_L) and adding the edge $v_2 \sim v_3$ in (V, T_R) gives a contradiction. Thus, (V, T_L) is a disjoint union of paths of at most one edge. Similarly, we have that (V, T_R) is a disjoint union of paths of at most one edge. Now if $v_1 \sim v_2$ in (V, T_L) and if $v_1 \sim v_2$ in (V, T_R) then deleting $v_1 \sim v_2$ in (V, T_R) gives a contradiction. Also, if $v_1 \sim v_2$ and $v_4 \sim v_5$ in (V, T_L) and if $v_2 \sim v_3$ and $v_5 \sim v_6$ in (V, T_R) then replacing $v_4 \sim v_5$ by $v_3 \sim v_4$ and $v_5 \sim v_6$ in (V, T_L) and deleting $v_2 \sim v_3$ and $v_5 \sim v_6$ in (V, T_R) gives a contradiction. Thus, each element of v is contained in an edge of T_L or of T_R, edges of T_L do not overlap, edges of T_R do not overlap, and there is at most occurrence of an edge of T_L overlapping an edge of T_R. Then $|\Pi_\Phi| = 2^{\lceil n/2 \rceil}$. □

A table of the bounds of lemma 18 is provided in table 1 below.

n	$2^{\lceil n/2 \rceil}$	$n!$
0	1	1
1	2	1
2	2	2
3	4	6
4	4	24
5	8	120
6	8	720
7	16	5040
8	16	40320
9	32	362880
10	32	3628800

Table 1: Values of the bounds of lemma 18.

Lemma 17 constrains the cardinality k of a parabolic double coset. Once k has been selected, the general procedure for obtaining the formula for the number of parabolic double cosets of cardinality k is as follows:

1. Use the inequalities of lemma 18 to constrain the possible values of $|V|$ for a maximal parabolic representation $\Phi = (V, A_L, A_R, T_L, T_R)$ with $|\Pi_\Phi| = k$.
2. Determine all maximal parabolic representations $\Phi = (V, A_L, A_R, T_L, T_R)$ with $|\Pi_\Phi| = k$.
3. Apply lemma 15 to each Φ, sum up the resulting formulas, and divide by k.

For step 2, the recommended approach is to first choose the value of $|V|$ (in the range given by step 1) and to then choose T_L and T_R such that $|\Pi_\Phi| = k$ and such that (V, T_L) and (V, T_R) are linear forests and such that every element of V is contained in some element of $T_L \cup T_R$. Once T_L and T_R have been chosen, let M_L and M_R be the collections of edges e such that Π_Φ is not closed under left or right multiplication by e (respectively). Then the valid choices of A_L and A_R are precisely those such that $T_L \subseteq A_L \subseteq T_L \cup M_L$ and $T_R \subseteq A_R \subseteq T_R \cup M_R$ and such that (V, A_L) and (V, T_R) are linear forests.

One has to take care to ensure that the resulting maximal parabolic representations are not isomorphic.

7 Examples

We now give examples of determining the formula for the number of parabolic double cosets of cardinalities $k = 2, 4, 6$. Note that there are no nontrivial parabolic double cosets of odd order by lemma 17.

7.1 $k = 2$

If $k = 2$ then Table 1 gives that $|V| = 2$. Also, lemma 18 gives that $A_L = T_L$ and $A_R = T_R$. If we write $V = \{a, b\}$ then there are three possibilities:

1. $A_L = T_L = \{\langle a, b\rangle\}$, $A_R = T_R = \varnothing$.
2. $A_L = T_L = \varnothing$, $A_R = T_R = \{\langle a, b\rangle\}$.
3. $A_L = T_L = \{\langle a, b\rangle\}$, $A_R = T_R = \{\langle a, b\rangle\}$.

In each case, $|\operatorname{Aut}(V, A_L, A_R, T_L, T_R)| = 2$. Also, cases 1 and 2 are symmetric. Then the general formula is

$$c_2(n) = \frac{(n-2)!}{4}\left(2 \cdot 2^1 (n-1)^2 (n-2) + 2^2 (n-1)^2\right) = (n-1)!(n-1)^2.$$

7.2 $k = 4$, the $|V| = 3$ case

If $k = 4$ then Table 1 gives that $|V| = 3, 4$. We only consider the case where $|V| = 3$. If we write $V = \{a, b, c\}$ then the only possibility for T_L and T_R (up to isomorphism) is $T_L = \{\langle a, b\rangle\}$ and $T_R = \{\langle b, c\rangle\}$. Note that $|\operatorname{Aut}(V, T_L, T_R)| = 1$ so $|\operatorname{Aut}(V, A_L, A_R, T_L, T_R)| = 1$. Then the contribution from the $|V| = 3$ case is

$$\begin{aligned}
&\frac{(n-3)!}{4} \sum_{\substack{A_L \supseteq T_L \text{ a} \\ \text{linear forest}}} \sum_{\substack{A_R \supseteq T_R \text{ a} \\ \text{linear forest}}} 2^{p_L + p_R - s_L - s_R} \frac{(n-2)!}{(n-p_L-2)!} \frac{(n-2)!}{(n-p_R-2)!} \\
&= (n-3)! \left(\sum_{\substack{A_L \supseteq T_L \text{ a} \\ \text{linear forest}}} \frac{(n-2)!}{(n-p_L-2)!} \right)^2 \\
&= (n-3)! \left((n-2)(n-3) + 2 \cdot (n-2)\right)^2 \\
&= (n-1)!(n-1)(n-2).
\end{aligned}$$

Note that this formula gives the correct result for $n = 3$ since the $|V| = 4$ case will not contribute anything when $n = 3$ (there are no injections from a set of size 4 to a set of size 3 so no φ_L exists).

7.3 $k = 6$

If $k = 6$ then table 1 gives that $|V| = 3, 4$. However, lemma 17 gives that $|\Pi_\Phi| \mid |H_L||H_R|$. Then $3 \mid |H_j|$ for some $j \in \{L, R\}$. Then T_j contains two adjacent edges $\langle a, b\rangle$ and $\langle b, c\rangle$ and $\Pi_\Phi = \mathfrak{S}_{\{a,b,c\}}$. This shows that $|V| = 3$. Then lemma 18 gives that $A_L = T_L$ and $A_R = T_R$. If we write $V = \{a, b, c\}$ then there are five possibilities (up to isomorphism):

1. $A_L = T_L = \{\langle a, b\rangle, \langle b, c\rangle\}$, $A_R = T_R = \varnothing$.
2. $A_L = T_L = \{\langle a, b\rangle, \langle b, c\rangle\}$, $A_R = T_R = \{\langle a, b\rangle\}$.
3. $A_L = T_L = \{\langle a, b\rangle, \langle b, c\rangle\}$, $A_R = T_R = \{\langle a, c\rangle\}$.
4. $A_L = T_L = \{\langle a, b\rangle, \langle b, c\rangle\}$, $A_R = T_R = \{\langle a, b\rangle, \langle a, c\rangle\}$.
5. $A_L = T_L = \{\langle a, b\rangle, \langle b, c\rangle\}$, $A_R = T_R = \{\langle a, b\rangle, \langle b, c\rangle\}$.
6. $A_L = T_L = \{\langle a, b\rangle\}$, $A_R = T_R = \{\langle a, b\rangle, \langle a, c\rangle\}$.
7. $A_L = T_L = \{\langle a, b\rangle\}$, $A_R = T_R = \{\langle a, b\rangle, \langle b, c\rangle\}$.
8. $A_L = T_L = \varnothing$, $A_R = T_R = \{\langle a, b\rangle, \langle b, c\rangle\}$.

By symmetry, the contribution from cases 6, 7, and 8 will be the same as the contribution from cases 1, 2, and 3. Then it suffices to consider cases 1, 2, 3, 4, and 5. In cases 1, 3, and 5, we have that $|\operatorname{Aut}(V, A_L, A_R, T_L, T_R)| = 2$ but in cases 2 and 4, we have that $|\operatorname{Aut}(V, A_L, A_R, T_L, T_R)| = 1$. Then the general formula is

$$\begin{aligned} c_6(n) &= \frac{(n-3)!}{6}(n-2)^2 \left(2\left(\frac{2^1}{2}(n-3)(n-4) + \frac{2^2}{1}(n-3) + \frac{2^2}{2}(n-3)\right) + \frac{2^2}{1} + \frac{2^2}{2}\right) \\ &= \frac{(n-2)!(n-2)}{3}\left((n-3)(n-4) + 4(n-3) + 2(n-3) + 3\right) \\ &= (n-2)!(n-2)(n^2 - n - 3)/3. \end{aligned}$$

8 Code

One of our goals at the beginning of this project was to implement the main formula in [1] for computing the number of parabolic double cosets in S_n. Over the past two quarters we have written some code which can be found at https://github.com/jir682/PDC. This has not only solidified our understanding of the theory, but has also allowed us collect data which has led to conjectures and theoretical results. Among other things, this code can:

- Compute the number of parabolic double cosets in S_n
- Find the minimal and maximal elements of a parabolic double coset
- Compute the lex-maximal presentation of a parabolic double coset (in interval form)
- Determine whether two permutations are related in Bruhat order
- Determine whether a Bruhat interval is a parabolic double coset
- Compute the rank and cardinality of a parabolic double coset
- Find all reduced expressions for a permutation
- Draw w-oceans

9 Future Goals

There is still much work to be done on counting parabolic double cosets by their rank and cardinality. We would like to eventually prove or disprove the following conjecture: For a fixed natural number k, the number of parabolic double cosets in S_n of rank $\binom{n}{2} - k$ is eventually constant as a function of n. In the future we also hope to explore other methods of enumeration, possibly based on minimal and maximal elements (since these are easy to compute).

References

[1] S. C. Billey, M. Konvalinka, T. K. Petersen, W. Slofstra, and B. E. Tenner. Parabolic double cosets in Coxeter groups. *Electr. J. Comb.*, 25:P1.23, 2018.

[2] A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups.* Springer Science+Business Media, Inc., 2005.