Number Theory and Noise

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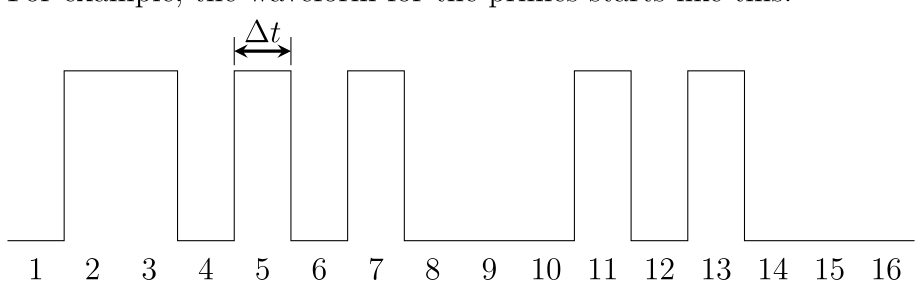
1 Introduction

The Number Theory and Noise project for Spring 2019 was led under faculty mentor Dr. Matthew Conroy and graduate mentor Kristine Hampton. The project has been ongoing since Spring 2016, and there are now over 350 sounds available for listening on the project website. This project investigates the possibilities arising from representing sets of positive integers as sound. A digital audio file is created from a given set A of positive integers by setting sample number i to a non-zero constant c for all i in the set. All other samples are set to zero.

For example, the waveform for the primes starts like this:



We use the standard CD-audio sampling rate of 44100 samples per second, so $\Delta t = \frac{1}{44100} = 0.0000226757...$ seconds.

For many sets, the result is what most people would describe as *noise*.

2 Rational Beatty Sequences

There are many sounds in the sound library produced for a variety of Beatty sequences. A Beatty sequence is a sequence of integers in which each term is found by taking the floor of a positive multiple of a positive irrational number. We can define Beatty sequences as $\{[\alpha n]\}$, where α is an irrational number. In order to better analyze the waveforms of Beatty sequences, we can use continued fractions to approximate α. Thus, we call them rational Beatty sequences. This leads to distinct properties of rational Beatty sequences that are defined below.

We know that an integer sequence $\{a_n\}$ is periodic if there is an integer P such that for all k in $\{a_n\}$, k + P is in the sequence.

Using continued fractions to approximate irrational constants in Beatty sequences give rise to **rational** and **periodic** Beatty sequences.

Let $\mathbf{B}$ be a rational Beatty sequence such that $\mathbf{B} = \{[\frac{a}{b}n]\}$, where $b < a$.

There are two properties of rational Beatty sequences:

1. For all r in the sequence, $r + a$ is in the sequence.

   Suppose $\alpha \in \mathbb{Q}$. Then $\alpha = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

   Let us define $\{a_n\}$ as $\{[\alpha n]\} = \{[\frac{a}{b}n]\}$.

   If $n = bk$, then $[\alpha n] = [\frac{abk}{b}] = [ak] = ak$.

   If r is in the sequence $\{[\frac{a}{b}n]\}$, then there is an $\hat{n}$ such that $r = [\frac{a}{b}\hat{n}]$.

   $[\frac{a}{b}(\hat{n} + b)] = [\frac{a}{b}\hat{n} + a] = [\frac{a}{b}\hat{n}] + a = r + a$.

   So, for all $r \in \{a_n\}$, $r + a \in \{a_n\}$.

2. Let $\mathbf{R} = \{[\frac{a}{b}n] : n \in \{0, 1, 2, ..., b-1\}\}$
   Then,

   $\{[\frac{a}{b}n]\} = \{j \in \mathbb{Z} : j \equiv r \pmod{a} \text{ for some } r \in \mathbf{R}\}$.

   (a) Suppose $x \in \mathbf{B}$. Then, $x = [\frac{a}{b}n]$ for some $n \in \mathbb{Z}_{\leq 0}$.

      If $0 \leq n < b$, then $x \in \mathbf{R}$.

      Suppose $n \geq b$.
      Then $n = vb + \widehat{n}$, where $v \in \mathbb{Z}$, $0 \leq \widehat{n} < b$.

Then, $x = [\frac{a}{b}n] = [\frac{a}{b}(vb + \widehat{n})] = [av + \frac{a}{b}\widehat{n}] = av + [\frac{a}{b}\widehat{n}]$, since $av \in \mathbb{Z}$.

So, $x \equiv [\frac{a}{b}\widehat{n}] \pmod{a}$, and $[\frac{a}{b}\widehat{n}] \in \mathbf{R}$.

(b) Suppose $x \equiv r \pmod{a}$ for some $r \in \mathbf{R}$.

Then, $x = r + wa$ for some $w \in \mathbb{Z}$.

Then, $r = [\frac{a}{b}i]$ for some i.

$x = wa + [\frac{a}{b}i] = [wa + \frac{a}{b}i] = [\frac{a}{b}(wb + i)] = [\frac{a}{b}n]$ where $n = wb + i$.

So, $x \in \mathbf{B}$.

However, not all sequences that are unions of a finite set of residue classes are rational Beatty sequences.

Consider the set $\mathbf{S}$ of integers congruent to 2, 3, or 9 mod 10.

This gives the sequence:

$$2, 3, 9, 12, 13, 19, 22, 23, 29, 32, 33, 39, ...$$

And the difference sequence:

$$1, 6, 3, 1, 6, 3, 1, 6, 3, ...$$

While the difference sequence is periodic, it is NOT a rational Beatty sequence.

Since a rational Beatty sequence is defined as $\{[\frac{a}{b}n]\}$, the difference between consecutive terms has two possibilites (there is a more in-depth proof by Dr. Conroy on the project website):

$$[\frac{a}{b}], \text{ or } [\frac{a}{b}] + 1$$

Since three different terms appear in the difference sequence of $\mathbf{S}$, the sequence $\mathbf{S}$ is not a rational Beatty sequence.

3 Sounds That Can Be Generated by a Sequence

What can we say about the totality of all sounds that could be generated from sequences using this specific procedure? In other words, what can be generalized about the sounds that are created from integer sequences? The procedure used in this project creates limitations on the waveform, for example, the output audio always consists of pulse waves which prevent the creation of "smooth" or sinusoidal waves. One method to better understand the limitations for the generated sounds is to reverse engineer the procedure - create an integer sequence that represents a given sound and then convert the sequence back into a sound and compare the difference. The reversed procedure demonstrates clear similarities between the original and reproduced audio, however, the reproduced sound is always more distorted.

4 Sums of Non-zero Squares

There are many sounds in the sound library that are produced from variations of sequences involving sums of nonzero squares.
The sum of 2 nonzero squares:

$$2, 5, 8, 10, 13, 17, 18, 20, 25, ...$$

The residue counts for this sequence show no uniformity however, modular arithmetic can be used to explain certain characteristics of the sequence.

Table 1.1 - each cell is the sum of the squares of the row and column modulo 4.

n	0	1	2	3
0	0	1	0	1
1	1	2	1	2
2	0	1	0	1
3	1	2	1	2

Table 1.1 demonstrates why there is a zero for n modulo 4 = 3

Table 1.2

n	n^2	$n^2 mod 8$
1	1	1
2	4	4
3	9	1
4	16	0
5	25	1
6	36	4

Similar to Table 1.1, Table 1.2 can be used to show that n^2 modulo 8 will always be 0, 1, or 4. Hence, the sum of two squares mod 8 can only be 0, 1, 2, 4, or 5. This explains why there is a 0 for n modulo 8 = 3, 6, 7.

Table 1.3

n	n^2	$n^2 mod 3$
1	1	1
2	4	1
3	9	0
4	16	1
5	25	1
6	36	0

able 1.4 shows that n^2 mod 3 will always be 0, or 1. Hence $(n^2 + n^2)$ mod 3 - the sum of two squares mod 3 can only be 0, 1, or 2. A probability table can be created using the pattern from Table 1.3 which demonstrates that the sum of 2 squares mod 3 will be 0 1/9Th of the time and 1 or 2 4/9Th's of the time each. As a result, the residue counts for the sum of 2 squares mod 3 is equal for the residues 1 and 2.

Further exploration into this sequence led to studying sub-sequence of this sequence - The sum of 2 nonzero squares in exactly n ways (the combination of these sub-sequences creates the original sequence. Although breaking down a sequence can help understand a sequence better, this specific example did not lead to new information.

The sum of 3 nonzero squares:

$$3, 6, 9, 11, 12, 14, 17, 19, 21, ...$$

The residue counts for this sequence show uniformity for every odd modulo - 3, 5, 7, 9, ... The pattern could potentially be explained using modular arithmetic similar to that used for the sum of 2 nonzero squares, however, it must be done using code rather than visual tables because of its 3-dimensional nature.

The sequence was explored further by breaking it into sub-sequences in hopes of understanding where the pattern for the uniformity and spectral lines begins to occur. To best observe the development of these characteristics, a sound file was generated for numbers that are the sum of 3 nonzero squares in exactly n ways where n is increasing geometrically by doubling every 2.2 seconds. The resulting spectrogram showed that the most prominent spectral lines occurred between 6.6 and 8.8 seconds (8 ways), however, the darkness of these lines changed throughout the rest of the audio file until the last 4.4 seconds (greater than and equal to 256 ways) which replicated the original sequences spectrogram.

5 Disjoint Sequences

When listening to a sound formed from an integer sequence it is often difficult to analyze it. An interesting way to understand the sound file is to break it down by producing sound files for disjoint sequences that when combined result in the original integer sequence. This allows us to not just understand the mathematical progression but hear the build up.

6 Abundant Numbers

A number n is abundant if the sum of divisors of n exceeds $2n$. Any integer multiple of an abundant number is also an abundant number. Splitting the abundant numbers into disjoint sequences required a look at primitive abundant numbers and perfect numbers.
Primitive abundant numbers are abundant numbers that have no abundant proper divisors. Perfect numbers are positive integers that are equal to the sum of their proper divisors.
The abundant numbers can be split into disjoint sequences in the following way: Generate a list of perfect or primitive abundant numbers. The first 10 are:

$$[6, 20, 28, 70, 88, 104, 272, 304, 368, 464]$$

For any number k, the kth disjoint sequence includes all the multiples of the kth element of the list that are not a multiple of the first $k-1$ elements of the list. For example, The first list would have 60, but the second list wouldn't. The final step was to convert these sequences into a sound file. The best way to do this to hear the auditory differences was to increase the values of k considered every 2 seconds. The first two seconds play the sequence for which k is 1, then the next two seconds play the numbers which satisfy the condition when k is either 1 or 2. The sound file slowly but surely starts to sound like abundant numbers and after $k = 6$ any changes are miniscule and are extremely subtle to the human ear.

7 Curved Digits

Let's look at all the primes that contain only curved digits. This sequence would include all the primes that don't have the numbers 1, 4, and 7, the non-curved digits.

$$2,\ 3,\ 5,\ 23,\ \dots$$

The sound file for this sequnece definitely has less noise than the sound file for the prime numbers sequence. The large gaps are caused due to the fact that $[100,000,\ 199,999]$, $[400,000,\ 499,999]$ and $[700,000,\ 799,999]$ are excluded from the file. This can be further generalized to just the sequence of numbers that contain only curved digits.

$$2,\ 3,\ 5,\ 6, 8,\ \dots$$

There is a strong similarity between this sound file and the curved primes with this file having less noise and more of a rythm to it.
Breaking this sound file down uses the fact that any integer sequence sounds exactly the same as the integer sequence's complement. The complement of this sequence in particular is the sequence of numbers that contain at least one straight, non-curved digit.

$$1,\ 4,\ 7, 10,\ 11$$

This complement can then be broken into disjoint subsequences. For any number k, the kth subsequence includes all the numbers where the kth digit from the right side is not curved and the first $k-1$ digits are curved. So, when k is 1, all the numbers have a 1, 4, or 7 as the rightmost digit. The sound file that actually displays the build up was made in a similar way to the abundant numbers: by increasing the interval for k every 2 seconds. It is evident that the sequences slowly approach the original sound. The fourth digit probably contributes the most to the pattern we hear in the original.