WXML Final Report: Rational Normal Curves

Faculty Mentor: Bianca Viray
Graduate Mentor: Chi-Yu Cheng
Team Members: Jocelin Liteanu, Leslie Feng

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1 Introduction

Curves in high dimensional space are generally difficult to visualize. In order to examine their properties, we need to reduce them to a lower dimension. The goal of this project is to explore the explicit ways of mapping rational normal curves in a field to quadratic curves in a subspace. In this report, we will elaborate on ways of reducing curves in $\mathbb{Q}^4$ to quadratics in two variables in $\mathbb{Q}^2$ by the way of projection.

1.1 Rational Normal Curves

Definition 1.1. The rational normal curve in $\mathbb{Q}^n$ is the image of the polynomial parameterization $\phi : \mathbb{Q} \to \mathbb{Q}^n$ given by

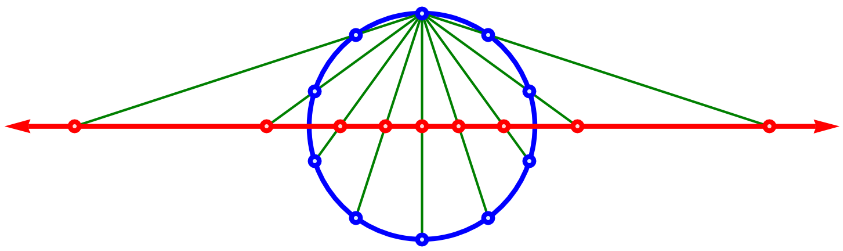
$$\phi(t) = (t, t^2, t^3, ..., t^n).$$

For the purpose of this project, we will simplify rational normal curves to conic curves, such as circles, ellipses, parabolas, and hyperbolas.

2 Projection away from a Point

Projecting the curve from a point reduces its ambient dimension by one, and the most basic example is from $\mathbb{Q}^2$ to $\mathbb{Q}$.

2.1 Example



In the picture above, the circle is projected from the topmost "fixed point" on the circle to the red line through its diameter. In order to represent the circle as various points on the line, we choose a moving point on its perimeter and compute the green line connecting the fixed and the moving points. The intersection of the green line with the red line is the projection of the circle as this specific fixed point. By varying the coordinates of the moving point, the projection of the entire circle is represented as a continuous segment on the red line.

However, not every rational normal curve is as simple. In general, the projection of these complicated curves are much harder to visualize, as illustrated by the example below, where the curve is reduced from in $\mathbb{Q}^4$ to $\mathbb{Q}^3$.

2.2 Example

Suppose that

$$C = \begin{cases} a^2 - b \\ ab - c \\ b^2 - ac \\ b^2 - d \end{cases} \subset \mathbb{Q}^4$$

Choose the point

$$p_1 = (1, 1, 1, 1),$$

away from which we want to project, and

$$p_2 = (\alpha, \beta, \gamma, \delta),$$

an arbitrary "moving" point on C. The line defined by these two points is given by:

$$\overline{p_1 p_2} = \langle(\alpha - 1)t + 1, (\beta - 1)t + 1, (\gamma - 1)t + 1, (\delta - 1)t + 1\rangle$$

What does the projection of C from p_1 look like on the plane $w = -1$? By direct computation,

$$\overline{p_1 p_2} \cap \{w = -1\} = \langle\frac{-2(\alpha - 1)}{\delta - 1} + 1, \frac{-2(\beta - 1)}{\delta - 1} + 1, \frac{-2(\gamma - 1)}{\delta - 1} + 1, -1\rangle$$

To understand the image, we must determine what polynomials vanish on the image. It comes down to solving the kernel of the following ring homomorphism:

$$\phi : \mathbb{Q}[x, y, z] \to \mathbb{Q}[a, b, c, d, \frac{1}{d-1}]/\langle I \rangle$$

$$x \mapsto \frac{-2(a-1)}{d-1} + 1$$

$$y \mapsto \frac{-2(b-1)}{d-1} + 1$$

$$z \mapsto \frac{-2(c-1)}{d-1} + 1$$

where I is the ideal that defines C. We compute the kernel on Macaulay2 using the following theorem:

Theorem 2.1. *Let I be an ideal of $\mathbb{Q}[x_1, ..., x_n]$, and $f \in \mathbb{Q}[x_1, ..., x_n]$ be a polynomial. Let $\mathbb{Q}[x_1, ..., x_n, \frac{1}{f}]$ be the ring extension of $\mathbb{Q}[x_1, ..., x_n]$ by inverting f, then there is a ring isomorphism*

$$\Phi : \mathbb{Q}[x_1, ..., x_n, \frac{1}{f}]/\langle I \rangle \longrightarrow \mathbb{Q}[x_1, ..., x_n, y]/(I + (yf - 1))$$

where

$$\Phi(x_i + I) = x_i + (I + (yf - 1))$$

$$\Phi(\frac{1}{f} + I) = y + (I + (yf - 1))$$

Proof. We will only prove the special case $\mathbb{Q}[t, \frac{1}{t}] \simeq \mathbb{Q}[t, s]/(ts - 1)$.
In this case $\Phi(t) = t + (ts - 1)$ and $\Phi(s) = s + (ts - 1)$. Define $\varphi : \mathbb{Q}[t, s] \to \mathbb{Q}[t, \frac{1}{t}]$ by

$$t \mapsto t$$

$$s \mapsto \frac{1}{t}.$$

Because $t \cdot \frac{1}{t} - 1 = 0$, φ factors through $\bar{\varphi} : \mathbb{Q}[t, s]/(ts - 1) \to \mathbb{Q}[t, \frac{1}{t}]$ where

$$t + (ts - 1) \mapsto t$$

and

$$s + (ts - 1) \mapsto \frac{1}{t}.$$

φ is the inverse of Φ. $\square$

Because of this, instead of mapping into $\mathbb{Q}[a,b,c,d,\frac{1}{d-1}]/\langle I\rangle$ we map into $\mathbb{Q}[a,b,c,d,e]/\langle I+(e(d-1)-1)\rangle$ in Macaulay2.

```
i18 : R=QQ[x,y,z];

i19 : S=QQ[a,b,c,d,e]/(a^2-b,c-a*b,b^2-a*c,d-b^2,e*(d-1)-1);

i20 : f=map(S,R,matrix{{-2*(a-1)*e+1,-2*(b-1)*e+1,-2*(c-1)*e+1}});

o20 : RingMap S <--- R

i21 : kernel f

              2                2                       2               2
o21 = ideal (y  - x*z - y*z + z  - x + y, x*y - x*z + z  + y - z - 1, x  - x*z + y*z + x - y - 1)
```

Therefore,

$$ker(\phi) = \langle y^2 - xz - yz - x + y, xy + xz + z^2 + y - z - 1, x^2 - xz + yz + x - y - 1\rangle.$$

3 Projection away from a Line

Based on the results from section 2, one might argue that reducing dimension of a curve can be attained through successive projections, dropping its dimension by one in every iteration. This method is infeasible when there does not exist any rational point on the curve. Under such circumstances, we need to consider reducing the dimension by more than one at once. The following approach diminishes the ambient space of a curve from $\mathbb{Q}^4$ to $\mathbb{Q}^2$:

1. Choose a rational line L that intersects the curve C at some points in a field possibly greater than $\mathbb{Q}$, and
2. Choose a 2D plane H that does not intersect L.

Together, any arbitrary moving point on the curve C and the line L define a moving 2D plane E_p, which should intersect H at a single point q. The projection away from L onto the plane H sends p to q. This procedure is demonstrated in the example below.

3.1 Example

$$C = \begin{cases} 3xz + 5yz - 1 \\ 3xy + 5y^2 - w \\ y - zw \\ 3x^2 + 5xy - w^2 \\ x - yw \\ xz - y^2 \end{cases} \subset \mathbb{Q}^4$$

Note that C does not contain any rational points, but it is defined in the extension field $\mathbb{Q}[\sqrt{37}]^4$. We can find a line with rational coefficients. If $(a_1 + b_1\sqrt{d}, a_2 + b_2\sqrt{d}, a_3 + b_3\sqrt{d}, a_4 + b_4\sqrt{d}) \in C$ where $a_i, b_i \in \mathbb{Q}$ then

$(a_1 - b_1\sqrt{d}, a_2 - b_2\sqrt{d}, \cdots, a_4 - b_4\sqrt{d}) = (\overline{A_1}, \overline{A_2}, \overline{A_3}, \overline{A_4})$. So subtracting $p_1 - \overline{p_1}$, we have $(2b_1\sqrt{d}, 2b_2\sqrt{d}, 2b_3\sqrt{d}, 2b_4\sqrt{d})$ where

$$p_1 = (A_1, \cdots A_4) \in \mathbb{Q}[\sqrt{37}]^4.$$

Therefore $\overline{p_1 p_2}$ must have rational coefficients.
We begin by finding the point p_1 on the curve.

$$p_1 : (\frac{31 - 5\sqrt{37}}{18}, \frac{-5 + \sqrt{37}}{6}, 1 + 0\sqrt{37}, \frac{\sqrt{37} - 5}{6})$$

Because L is defined by equations with rational (integral) coefficients, the conjugate of p_1, p_2, written as

$$p_2 : (\frac{31 + 5\sqrt{37}}{18}, \frac{-5 - \sqrt{37}}{6}, 1 - 0\sqrt{37}, \frac{-\sqrt{37} - 5}{6})$$

is also a point on the curve. $\overline{p_1 p_2} = \langle \frac{1}{3} - \frac{5}{3}t, t, 1, t\rangle$ the line away from which we want to project. Given $p = (\alpha, \beta, \gamma, \delta) \in C$, we have the plane

$$E_p := \langle t\alpha - \frac{t}{3} - \frac{5}{3}s + \frac{1}{3}, s + t\alpha, 1 + \gamma t - t, s + t\beta\rangle$$

Intersecting it with the planes $y = 0$ and $x = 0$,

$$E_p \cap \{x = y = 0\} = (0, 0, \frac{-3\alpha - 5\beta + \gamma}{1 - 3\alpha - 5\beta}, \frac{\delta - \beta}{1 - 3\alpha - 5\beta})$$

The final projection is defined by the map

$$\phi : \mathbb{Q}[x, y] \to \mathbb{Q}[a, b, c, d, \frac{1}{1 - 3a - 5b}]/\langle I\rangle$$

where

$$x \mapsto \frac{-3a - 5b + c}{1 - 3a - 5b}$$
$$y \mapsto \frac{d - b}{1 - 3a - 5b}$$

In order to find the kernel mentioned above using Macaulay2, again we rely on theorem 2.1. The screenshot below shows the implementation:

```
i1: R=QQ[x,y];

i2: S=QQ[a,b,c,d,e]/(3*a*c + 5*b*c - 1,  3*a*b + 5*b^2 - d,  3*a^2 + 5*a*b - d^2,
        b - c*d, a - b*d,  a*c - b^2, e*(1-3*a-5*b)-1);

i3: f=map(S,R,matrix{{(-3*a-5*b+c)*e,(d-b)*e}});

o3: RingMap S <--- R

i4: kernel f

                      2
o4 = ideal(5x*y - 3y  - 5y + 1)
```

Thus $ker(\phi) = 5xy - 3y^2 - 5y + 1$.

4 Future Goals

1. Prove theorem 2.1 in full generality;
2. Explore different types of spaces from which we could project a curve, e.g. projecting from a plane;
3. Figure out a systematic approach to projecting regardless of the dimension of the ambient space.

References

[1] D. Cox et al. Projective Algebraic Geometry. *Ideals, Varieties, and Algorithms: An introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer, New York, 3rd Edition, 2007.