# WXML FINAL REPORT
# TRIPLY PERIODIC POLYHEDRAL SURFACES

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Abstract. We explored plane sections of triply periodic surfaces $\{p, q \mid n\}$ comprised of regular $p$-sided polygons with $q$ such polygons meeting at each vertex and $n$-gonal holes. In particular, we provide plane sections of the surfaces $\{3, 8 \mid 3\}$, $\{3, 12 \mid 3\}$ and $\{4, 6 \mid 4\}$.

## 1. Introduction

A *triply periodic polyhedral surface* is an infinite polyhedron that repeats along three independent directions in $\mathbb{R}^3$. Given a triply periodic polyhedral surface $\Pi$, what kind of patterns result from the intersection of $\Pi$ with a plane?

**Motivations.** One reason for studying plane sections of triply periodic polyhedral surfaces concerns the theory of quasiperiodic functions on the plane and electron transport, in particular particle diffusion in magnetic fields (see [3]).

It is known that certain triply periodic functions give rise to fractals (see the introduction and Section 2 in [1]). De Leo in [1] analyzes a particular triply periodic function associated to the triply periodic polyhedral surface composed of truncated octahedra glued along its triangular faces.

## 2. The Main Examples

We studied the Octa-4 $\{3, 8 \mid 3\}$, the Octa-8 $\{3, 12 \mid 3\}$ and the Cube-6 $\{4, 6 \mid 4\}$ triply periodic surfaces. To study these surfaces, we look at their *fundamental region (or piece)*, i.e., the minimal connected piece of the polyhedron that, under the three translations, spans the surface. See Figures 2.1, 2.2 and 2.3 for fundamental regions of the Octa-4, Octa-8 and Cube-6 triply periodic surfaces, respectively. The fundamental region for Octa-4 consist of *central* octrahedra (colored brown) with other octahedra, which we refer to as *handles*, glued to four non-adjacent faces of the central octahedron's faces. The fundamental region for Octa-8 consists of half-octahedrons, which we also refer to as handles, glued to the eight faces of a central (hidden) octahedron. Finally, the fundamental region for the Cube-6 surface is a central cube with six cube handles attached to the faces of the central cube. (Note that, once polyhedra are glued along faces, the glued face is then effaced and the remaining faces are considered part of the triply periodic surface.)

See Figure 2.4 for a 3D-print of the surface, as well as http://www.wxml.math.washington.edu/?p=555 for more pictures.

Translations of the fundamental regions by three particular independent directions in $\mathbb{R}^3$ result in an infinite polyhedral surface. Thus, forming the quotient space

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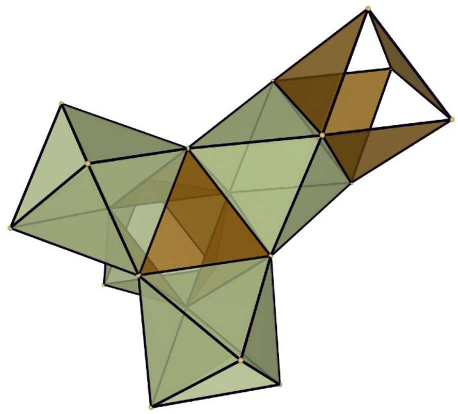


Figure 2.1. The fundamental region of the Octa-4 surface, composed of 32 triangle faces with eight triangles meeting at each of the 12 vertices.

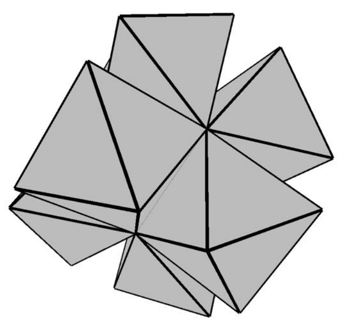


Figure 2.2. Part of the fundamental region of the Octa-8 surface, composed of 24 triangles with 12 triangles meeting at each of the six vertices. Taking half of each octahedron yields the fundamental region.

of each fundamental region by some rank-3 lattice in $\mathbb{R}^3$ results in an abstract, orientable, compact surface. Applying Euler's formula $V-E+F=2-2g$ where $V, E$ and $F$ denote the number of vertices, edges, and faces of the polyhedral surface, respectively and $g$ is the genus, we find that the Octa-4 and Cube-6 surfaces have genus three, while the Octa-8 surface has genus four.

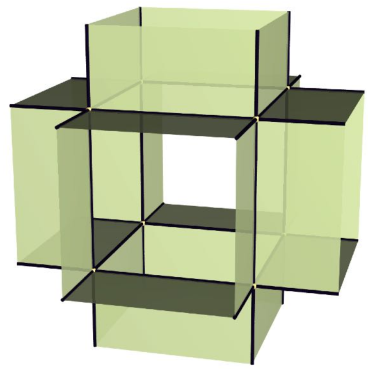


FIGURE 2.3. The fundamental region of the Cube-6 surface, composed of 12 square faces with six squares meeting at each of the eight vertices.

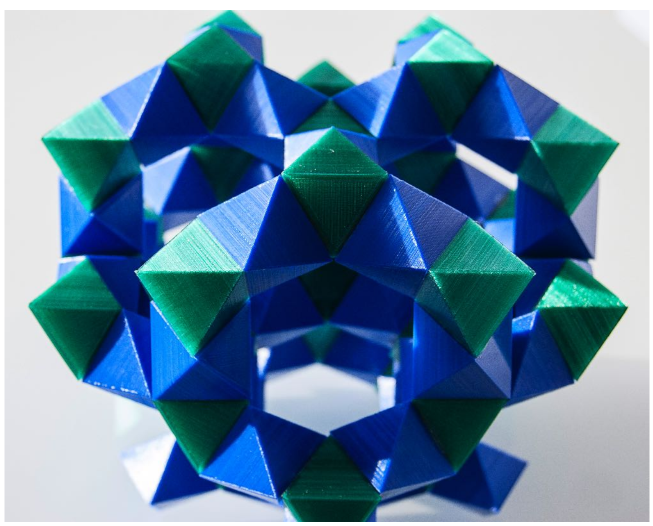


FIGURE 2.4. The triply periodic polyhedral surface Octa-4, obtained by gluing several fundamental regions.

## 3. Our Findings

**Vertices and Translation Directions.** Before computing plane sections, we first determined the vertices of the fundamental regions of the surfaces described above, as well as their translation directions.

Octa-4: Place a center octahedron at the origin with its six vertices at a distance of $3/4$ away from the origin along the six coordinate axes. The four handle octahedra are attached so that their vertices lie at a distance of $3/4$ away from the points $(1, 1, 1), (-1, -1, 1), (1, -1, -1)$ and $(-1, 1, -1)$. In other words, octahedra formed from unit cubes which tile $\mathbb{R}^3$ are the handle octahedra of Octa-4. See Figure 3.1 (this figure used cubes of length $4/3$, rather than unit cubes). The translation directions are then seen to be the three diagonal directions

$$\langle 2,2,0\rangle, \quad \langle 2,0,2\rangle, \quad \langle 0,2,2\rangle.$$

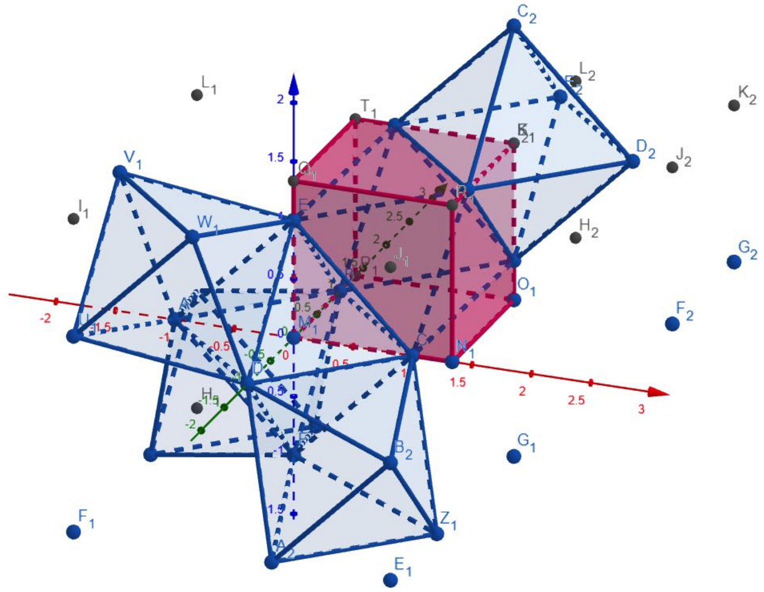



FIGURE 3.1. Octa-4 formed by taking points 3/4 away along the edges of a cube.

<u>Octa-8</u>: This time, place a unit octahedron at the origin with vertices $\left(0,0,\pm\frac{\sqrt{2}}{2}\right)$ and $\left(\pm\frac{1}{2},\pm\frac{1}{2},0\right)$. The vectors that point to the vertices of the unit octahedron represent the three directions where the surface repeats. The fundamental region has length, width, and height $\frac{4\sqrt{2}}{3}$, rotated about the $z$-axis. Thus, the independent translations that preserve the surface are the vectors

$$\left\langle \frac{4}{3},\frac{4}{3},0\right\rangle, \quad \left\langle \frac{4}{3},-\frac{4}{3},0\right\rangle, \quad \left\langle 0,0,\frac{4\sqrt{2}}{3}\right\rangle.$$

The vertices of Octa-8 that lie in the fundamental unit are $(\pm\frac{1}{2},\pm\frac{5}{6},\pm\frac{2\sqrt{2}}{3})$, $(\pm\frac{5}{6},\pm\frac{1}{2},\pm\frac{2\sqrt{2}}{3})$, $(0,0,\pm\frac{\sqrt{2}}{2})$, $(\pm\frac{1}{2},\pm\frac{1}{2},0)$, $(0,\pm\frac{4}{3},\pm\frac{\sqrt{2}}{6})$, and $(\pm\frac{4}{3},0,\pm\frac{\sqrt{2}}{6})$. However, some of these lie on the boundary (unlike $(4, 6, 6)$). Therefore, for integers $i, j$ and $k$, the coordinates of all the vertices of Octa-8 are

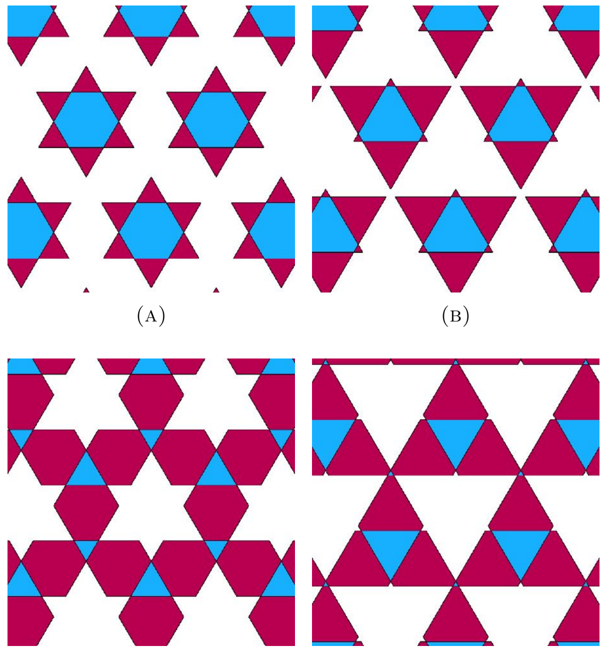
$$\left(\frac{1}{2}, \pm\frac{5}{6}, \frac{2\sqrt{2}}{3}\right) + \left(\frac{4}{3}(i+j), \frac{4}{3}(i-j), \frac{4\sqrt{2}}{3}k\right)$$
$$\left(\frac{5}{6}, \pm\frac{1}{2}, \frac{2\sqrt{2}}{3}\right) + \left(\frac{4}{3}(i+j), \frac{4}{3}(i-j), \frac{4\sqrt{2}}{3}k\right)$$
$$\left(0, 0, \pm\frac{\sqrt{2}}{2}\right) + \left(\frac{4}{3}(i+j), \frac{4}{3}(i-j), \frac{4\sqrt{2}}{3}k\right)$$
$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, 0\right) + \left(\frac{4}{3}(i+j), \frac{4}{3}(i-j), \frac{4\sqrt{2}}{3}k\right)$$
$$\left(\frac{4}{3}, 0, \pm\frac{\sqrt{2}}{2}\right) + \left(\frac{4}{3}(i+j), \frac{4}{3}(i-j), \frac{4\sqrt{2}}{3}k\right)$$

Cube-6: Place the unit cube centered at the origin with vertices $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Since the fundamental region of the Cube-6 surface has side length 2 in the $x$, $y$, and $z$ directions, the translation vectors that preserve the structure are

$$\langle 2, 0, 0\rangle, \quad \langle 0, 2, 0\rangle, \quad \langle 0, 0, 2\rangle.$$

Combining the coordinates of the unit cube, which happen to be all vertices that lie in the fundamental unit, with the three translations, we obtain the vertices $\left(2i \pm \frac{1}{2}, 2j \pm \frac{1}{2}, 2k \pm \frac{1}{2}\right)$, for integers $i, j$ and $k$. This describes all vertices of the Cube-6 surface.

**Plane Sections.** We computed the intersection of a plane with each of the three triply periodic surfaces. See Figure 3.2 for plane sections of Cube 6, Figures 3.3 and 3.4 for plane sections of Octa-4, and Figure 3.5 for plane sections of Octa-8. Blue-colored polygons denote center polyhedra, while magenta-colored polygons denote handle polyhedra. (The code generating these plane sections is written in Java script and is available upon request. Full .gif files are also available.)



(A) (B) (C) (D)

FIGURE 3.2. Various plane section of the Cube-6 surface the plane with normal vector $\langle 1, 1, 0\rangle$.

## 4. Conclusion and Further Exploration

**Other Periodic Surfaces.** We investigated other candidates for triply periodic polyhedral surfaces. Lee [6] proves that the Octa-4 surface is realizable as the Fermat quintic, a compact Riemann surface of genus three with a cyclic eightfold cover of the sphere branching over three points. This is given by the defining algebraic equation $\{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x^4 + y^4 = z^4\}$. However, it is known that there is another Riemann surface with a similar property, having defining equation $\{(x, y) \in \mathbb{C}^2 : y^2 = x^8 - 1\}$ (see, e.g., Equation (5.6) in Table 5.3 in [5]). We believed that it may be possible to create polyhedralization of this other surface by gluing truncated cubes along their triangular faces (see Figure 4.1). Unfortunately, after printing several truncated cubes, we were unable to see any repeating patterns. Perhaps this Riemann surface cannot be realized as a triply periodic polyhedron?

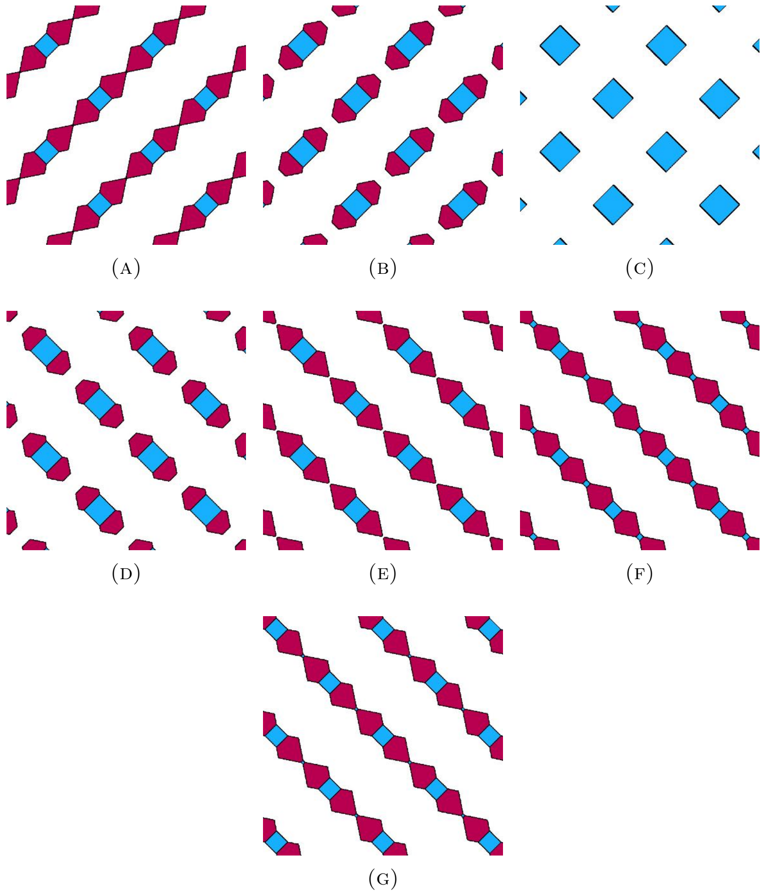



FIGURE 3.3. Various plane section of the Octa-4 surface through the plane with normal vector $\langle 1, 1, 0\rangle$.

We also considered attaching square prisms along the square faces of a rhombicuboctahedron (see Figure 4.2) that are parallel to the coordinate axes. Although this surface is not regular in the sense that every face is congruent, it will create a triply periodic surface, whose translation directions are the coordinate axes. We would be interested in studying these semi-regular triply periodic surfaces.

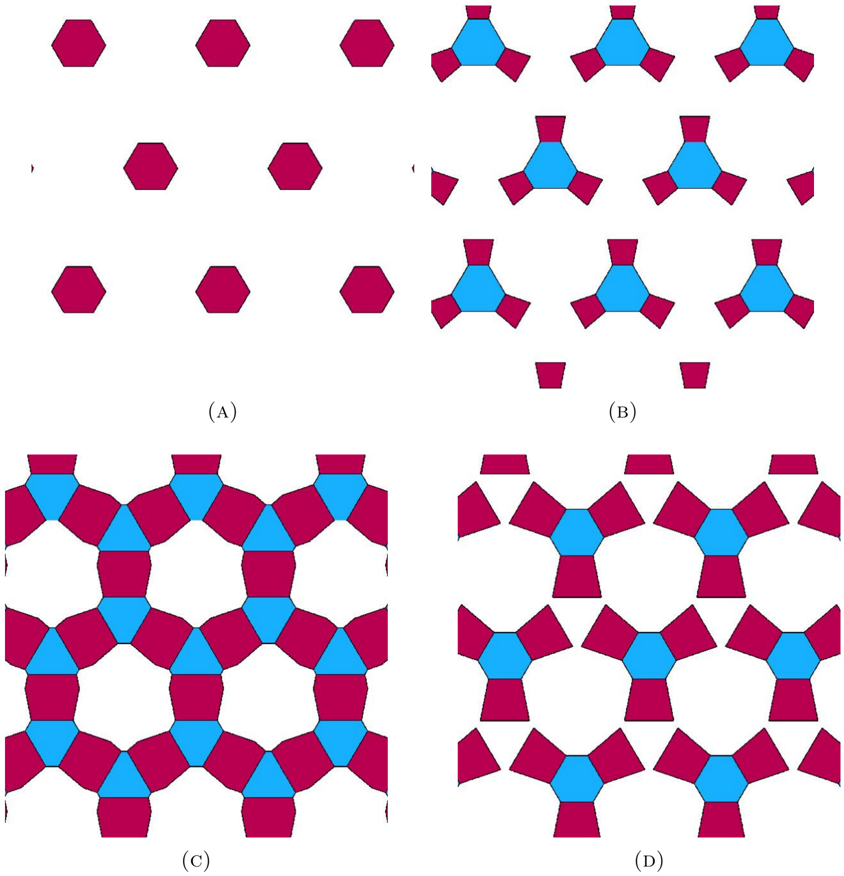



FIGURE 3.4. Various plane section of the Octa-4 surface through a handle of the surface. The plane has normal vector $\langle 1, 1, 0 \rangle$.

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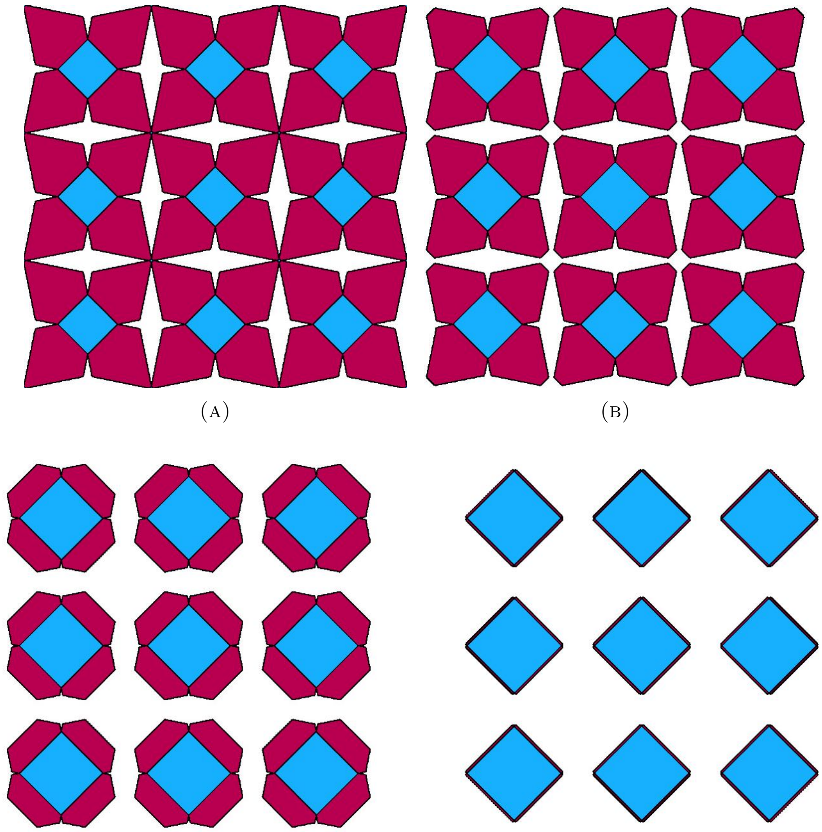

(A) (B) (C) (D)

Figure 3.5. Various plane section of the Octa-8 surface through the plane with normal vector $\langle 0, 0, 1 \rangle$.

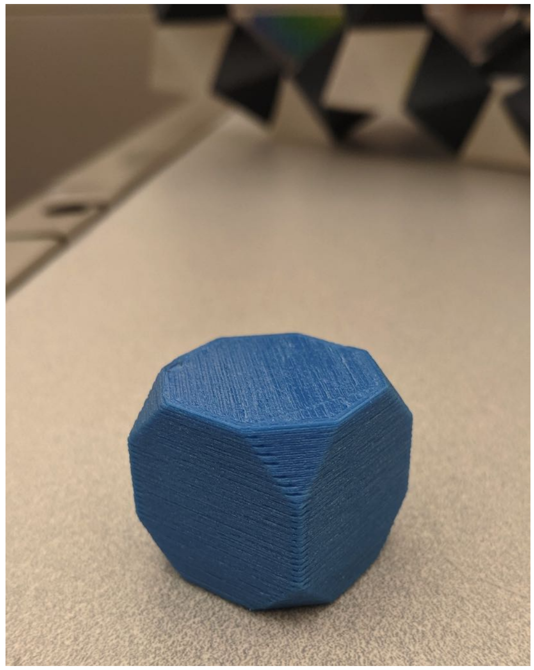

FIGURE 4.1. Truncated cube, formed from a cube by intersecting it with planes that cut the vertices off the cube leaving regular octagonal faces and equilateral triangular faces. This truncated cube was printed at the WXML.

FIGURE 4.2. A rhombicuboctahedron, comprised of eight triangles and eighteen squares. This rhombicuboctahedron was also printed at the WXML.