WXMLE: LOCATING DISJUNCTIVE WALSH INEQUALITIES IN THE SNIPEP

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1. Introduction

1.1. Introduction to SNIEP.

Definition 1.1 (NIEP[1]).

The Nonnegative Inverse Eigenvalue Problem (NIEP) asks which collections of
n complex numbers (counting multiplicities) occur as the eigenvalues of an n-by-n
matrix, all of whose entries are nonnegative real numbers.

Based on this idea, the Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP)
is defined as the following:

Definition 1.2 (SNIEP[1]).

Determine which sets of n real numbers occur as the spectrum of an n-by-n sym-
metric nonnegative matrix.

Many theorems\(^1\) have discussed the solutions of SNIEP problem, but their descrip-
tions only contain some subsets of the desired spectrum. Therefore, it is important
to determine the inclusion or exclusion relationships between these subsets. And this
is the research question to locate Walsh Inequalities in the SNIEP map.

1.2. Introduction to Walsh Matrices and Inequalities.

We define a \(2^n \times 2^n\) Walsh matrix recursively as follows:

\[
H_0 = \begin{bmatrix} 1 \end{bmatrix},
\]

\[
H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, \quad n \geq 1.
\]

For example, the \(4 \times 4\) Walsh matrix is

\[
H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.
\]

With this definition of a Walsh matrix, the reader should easily be able to verify
that Walsh matrices have these properties:

\(^1\)To see all the theorems, please check MarijuĂĄn, Pisonero, & Soto. (2017). A map of suffi-
cient conditions for the symmetric nonnegative inverse eigenvalue problem. Linear Algebra and Its
Applications, 530, 344-365.
(a) Symmetry
(b) $H_n^{-1} = 2^{-n}H_n$
(c) $tr(H_n) = 0$ for $n \geq 0$
(d) $e_1^T H_n = e^T$
(e) $H_n e_1 = e$
(f) $e_1^T H_n = n e_1$

With the Walsh matrix, we define the **Walsh inequalities** as the set of inequalities that satisfy

$$H_n v \geq 0 \quad v \in \mathbb{R}_{2^n}^2$$

where $\mathbb{R}_{2^n}^2 = \{x \in \mathbb{R}^{2^n} : x_1 \geq x_2 \geq \cdots \geq x_{2^n}\}$.

For example, the Walsh inequalities constructed by the $4 \times 4$ Walsh matrix is the set of inequalities

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 & \geq 0 \\
    x_1 - x_2 + x_3 - x_4 & \geq 0 \\
    x_1 + x_2 - x_3 - x_4 & \geq 0 \\
    x_1 - x_2 - x_3 + x_4 & \geq 0
\end{align*}
\]

where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}_4^4$. Since $x \in \mathbb{R}_4^4$, we further have the property that $x_1 \geq x_2 \geq x_3 \geq x_4$. As we will see in the next section, that property will be noted as a very powerful fact to be used in proving the SNEIP for $4 \times 4$ matrices.

2. **Proving the SNEIP for $n = 4$**

Let

$$\mathbb{R}_4^4 = \{x \in \mathbb{R}^4 : x_1 \geq x_2 \geq x_3 \geq x_4\}.$$ 

Construct

$$S_1 = \{x \in \mathbb{R}_4^4 : x_1 + x_2 + x_3 + x_4 \geq 0 \land x_1 \geq -x_4\}$$
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and

\[ S_2 = A \cup B \]

\[
\begin{cases}
  x_1 + x_2 + x_3 + x_4 \geq 0 \\
  x_1 - x_2 + x_3 - x_4 \geq 0 \\
  x_1 + x_2 - x_3 - x_4 \geq 0 \\
  x_1 - x_2 - x_3 + x_4 \geq 0
\end{cases}
\cup
\begin{cases}
  x_1 + x_4 \geq 0 \\
  x_1 - x_4 \geq 0 \\
  x_2 + x_3 \geq 0 \\
  x_2 - x_3 \geq 0
\end{cases}
\] \quad x \in \mathbb{R}_4^4

We will show that \( S_1 = S_2 \) by showing that the sets are subsets of each other. We first show the easier direction of showing that \( S_2 \subseteq S_1 \). Let \( x \in S_2 \). Then since \( x_1 + x_2 + x_3 + x_4 \geq 0 \) and \( x_1 - x_2 - x_3 + x_4 \geq 0 \), it follows that \( x_1 \geq -x_4 \). So we have that \( x \in S_1 \), and consequently, \( S_2 \subseteq S_1 \). Now we will show that \( S_1 \subseteq S_2 \). So let \( x \in S_1 \). By the sufficient conditions given by Suleimenova, it follows that \( x_1 \geq |x_j| \) for \( 1 \leq j \leq 4 \). It suffices to show that \( X \in A \) or \( x \in B \). It easily follows that the first three inequalities for \( A \) are held. Now for the fourth inequality, we will do an argument by cases.

**Case 1:** \( x_1 - x_2 - x_3 + x_4 \geq 0 \). Then all four inequalities of \( A \) are held and \( x \in A \), and consequently, \( S_1 \subseteq S_2 \).

**Case 2:** \( x_1 - x_2 - x_3 + x_4 < 0 \). Then we will show that \( x \in B \). We know that \( x_1 + x_4 \geq 0 \) since \( x \in S_1 \). Furthermore, since \( x \in S_1 \subseteq \mathbb{R}_4^4 \) and by the Suleimenova condition, the inequality \( x_1 - x_4 \geq 0 \) holds trivially. Then from \( x_1 - x_2 - x_3 + x_4 < 0 \), it follows that \( x_2 + x_3 > x_1 + x_4 \). Since \( x_1 + x_4 \geq 0 \) from the first inequality from \( B \), it follows that \( x_2 + x_3 \geq 0 \).

3. MATLAB Codes to Locate Walsh Matrices

Consider the research question now. It will cost a lot to do algebra work to locate the sets of spectrum from Walsh Inequalities. This characteristic is easy to see when \( n = 4 \) as the above. So the idea is to use coding. MATLAB, as an excellent tool in matrices multiplication, is a good option. As the starting try, we try to determine the relationship between spectrums from Walsh matrices and Fiedler, when \( n = 4 \) in MATLAB.
Theorem 3.1 (Fiedler[2], 1974).

Let \( \Lambda = \{\lambda_0, \lambda_1, ..., \lambda_n\} \) with \( \lambda_i \geq \lambda_{i+1} \) for \( i = 0, ..., n - 1 \). If

\[
\lambda_0 + \lambda_n + \sum_{\lambda \in \Lambda} \lambda \geq \frac{1}{2} \sum_{1 \leq i \leq n-1} |\lambda_i + \lambda_{n-i}|,
\]

then \( \Lambda \) is symmetrically realizable.

3.1. Sample code for \( n = 4 \). The general idea is using MATLAB to define two functions that determine whether a set of random values satisfies Walsh matrices or Fiedler, or both of them. By repeating this process for over a large number of tries, we can get a hypothesis. Therefore, it will curtail the workload in pure algebra proofs.

The following is a sample code for \( n = 4 \).

```matlab
1  %% Setting up
2  range = 4; % how many eigenvalues do I want
3  times = 10000; % loops to figure out random
4      % sets of eigenvalues
5
6  % in walsh and in Fiedler
7  % So x(4) is our x1, the biggest eigenvalue
8  % x(3) is our x2, x(2) is our x3, and x(1) is our x4,
9  for i = 1: times
10     x0 = 2*(rand(4, 1)-0.5);
11     x = sort(x0);
12     if walshTest(x) == true && fiedlerTest(x) == true
13         disp("in walsh and in Fiedler")
14         x
15         break
16     end
17  end
18
19  % not in walsh and in Fiedler
20  for i = 1: times
21     x0 = 2*(rand(4, 1)-0.5);
22     x = sort(x0);
23     if walshTest(x) == false && fiedlerTest(x) == true
24         disp("in walsh and in Fiedler")
25     end
```
3.2. A Short Proof. Based on the hypothesis from previous codes, a short proof is attached below to show a set of four values satisfying the Walsh inequalities is also in the Fiedler inequality.
Suppose that \( x_1, x_2, x_3, x_4 \) satisfy the Walsh inequalities:

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &\geq 0 \\
    x_1 - x_2 + x_3 - x_4 &\geq 0 \\
    x_1 + x_2 - x_3 - x_4 &\geq 0 \\
    x_1 - x_2 - x_3 + x_4 &\geq 0
\end{align*}
\]

We want to show that they satisfy the Fiedler inequality:

\[
2x_1 + 2x_4 + x_2 + x_3 \geq |x_2 + x_3|.
\]

By definition of the absolute value it’s sufficient (and in fact necessary) to show that the left-hand side is bigger than or equal to both \((x_2 + x_3)\) and \(-(x_2 + x_3)\). Let’s first show the “\(\geq (x_2 + x_3)\)” case. Adding the first and the fourth Walsh inequalities gives:

\[
2x_1 + 2x_4 \geq 0.
\]

Adding \(x_2 + x_3\) to both sides gives us our first case:

\[
2x_1 + 2x_4 + x_2 + x_3 \geq (x_2 + x_3).
\]

Now let’s prove the “\(\geq -(x_2 + x_3)\)” case. Multiply through the first Walsh inequality by 2 to get that:

\[
2x_1 + 2x_2 + 2x_3 + 2x_4 \geq 0.
\]

Carrying one copy of \(x_2\) and \(x_3\) to the other side gives us this other case:

\[
2x_1 + 2x_4 + x_2 + x_3 \geq -(x_2 + x_3).
\]

Hence \(x_1, x_2, x_3, x_4\) are in Fiedler:

\[
2x_1 + 2x_4 + x_2 + x_3 \geq |x_2 + x_3|.
\]

References
