Nontransitive Dice and Arrow’s Theorem

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Chapter 1

Dice proofs

1.1 Background and Introduction

Last quarter, the last group has proved that when A, B, and C are three dice, and p is the probability that A beats B, q is the probability that B beats C, and r is the probability that C beats A, then $1 \leqslant p+q+r \leqslant 2$.

This inequality tells us that if p, q, and r are equal, then they are at most $\frac{2}{3}$. But we cannot find a set of dice where the probabilities are all equal to $\frac{2}{3}$. In Section 1.2 we present the proof that p, q, and r cannot all be $\frac{2}{3}$ at the same time. This proof shows that the map from dice to voting system is not onto, and we have a crucial observation about dice and voting theory. We further investigate properties of three dice in Section 1.3 and a property of four dice in Section 1.4. Then we infer a general property in dice, which is formally proved by induction in Section 1.5.

Note that in all proofs, there is no tie between dice.

1.2 Limitation of Translating Voting System to Dice

In this section, we start with the proof that p, q, and r cannot all be $\frac{2}{3}$, and conclude that some cases in voting system cannot be shown in dice.

Proposition: p, q, and r cannot all be $\frac{2}{3}$.

Proof:

Suppose A, B, and C are three dice, and a, b, and c represent numbers on A, B, and C.
Let u be the possibility of $a>b>c$.
Let v be the possibility of $c>a>b$.
Let w be the possibility of $b>c>a$.
Let x be the possibility of $c>b>a$.
Let y be the possibility of $b>a>c$.
Let z be the possibility of $a>c>b$.
Let $p=x+y+z$.
Suppose that the possibilities of A beating B, B beating C, and C beating A are all $\frac{2}{3}$.
Since the possibility of A beating B is $\frac{2}{3}$, $u+v+z=\frac{2}{3}$ and $w+x+y=\frac{1}{3}$.

Since the possibility of B beating C is $\frac{2}{3}$, u+w+y=$\frac{2}{3}$ and v+x+z=$\frac{1}{3}$.

Since the possibility of C beating A is $\frac{2}{3}$, v+w+x=$\frac{2}{3}$ and u+y+z=$\frac{1}{3}$.

Because u+v+z=$\frac{2}{3}$, u+w+y=$\frac{2}{3}$, and v+w+x=$\frac{2}{3}$, then 2u+2v+2w+x+y+z=2.

Because u+v+z=$\frac{2}{3}$ and w+x+y=$\frac{1}{3}$, then u+v+w+x+y+z=1.

Since 2u+2v+2w+x+y+z=2 and u+v+w+x+y+z=1, then u+v+w=1.

Suppose p=0.

Then x+y+z=0, which means that x=y=z=0.

Since w+x+y=$\frac{1}{3}$, v+x+z=$\frac{1}{3}$, and u+y+z=$\frac{1}{3}$, we have w=v=u=$\frac{1}{3}$.

Because y=z=0, which means that there is no such case that a is larger than c and smaller than b, and there is no such case that a is larger than b and smaller than c.

Then a>b>c cannot happen, and so u=0$\neq\frac{1}{3}$, which is a contradiction.

So p cannot be 0.

Since p is possibility which cannot be negative, then p has to be larger than 0.

Since u+v+w+x+y+z=1 and p>0, if u+v+w=1, then1+p≠1, which is a contradiction.

Therefore, our assumption is wrong, which means that it is impossible that possibilities of A beating B, B beating C, and C beating A are all $\frac{2}{3}$.

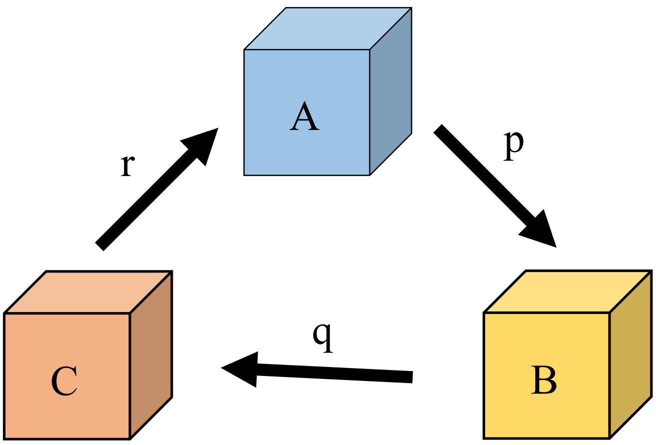
Q.E.D

This instance is valid in voting system but not in dice, which means that there exists a case that the voting system cannot be represented by dice. Then we can say that the map from dice to voting system is not onto.

To explain why dice cannot show all possibilities in voting systems, we observe that all possible outcomes of voters exist in voting theory, but we can only pick the labelling of the dice sides in dice, and this restricts our set of outcomes.

1.3 Properties of Three Dice

Based on the inequality 1⩽p+q+r⩽2, we discover the properties of three dice when the sum is one and when the sum is two.



Proposition: If p+q+r=1, then at least one of p, q, r must equal 0.

Proof:

Suppose a, b, and c represent numbers on dice A, B, and C.
Let u be the probability of a>b>c.
Let v be the probability of c>a>b.
Let w be the probability of b>c>a.
Let x be the probability of c>b>a.
Let y be the probability of b>a>c.
Let z be the probability of a>c>b.
Then, p=u+z+v, q=w+y+u, and r=v+x+w.
Suppose p+q+r=1.
Then, (u+z+v)+(w+y+u)+(v+x+w)=1.
We also know that u+v+w+x+y+z=1.
We get that u=v=w=0 and x+y+z=1.
Consider the case when the largest side of all dice is on dice A, and that side is rolled.
Then, the only possible outcomes are a>b>c or a>c>b, which have probabilities u and z.
Since u=0, then a>c>b must happen.
Since A is independent of B and C, this means that 1-q, the probability that C beats B, must equal 1.
The cases where B or C have the largest side follow the same logic, by symmetric argument.
Thus, if p+q+r=1, then at least one of p, q, r must equal 0.

Q.E.D

Proposition: If p+q+r=2, then at least one of p, q, r must equal 1.

Proof:

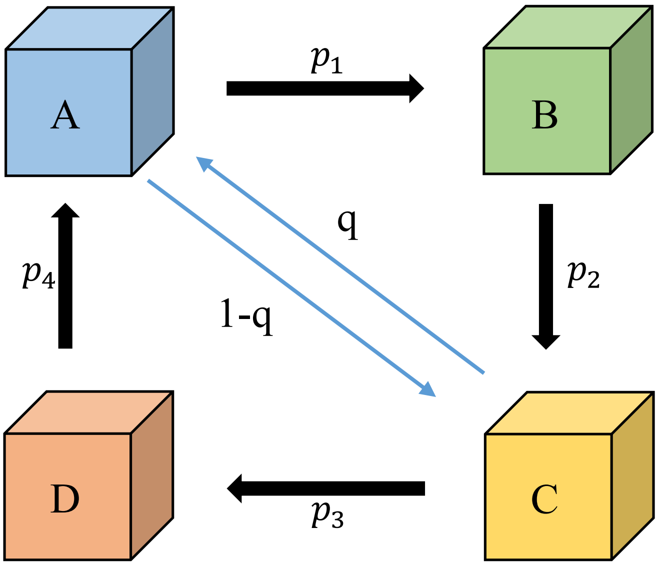
Suppose a, b, and c represent numbers in sets A, B, and C.
Let u be the probability of a>b>c.
Let v be the probability of c>a>b.
Let w be the probability of b>c>a.
Let x be the probability of c>b>a.
Let y be the probability of b>a>c.
Let z be the probability of a>c>b.
Then, p=u+z+v, q=w+y+u, and r=v+x+w.
Suppose p+q+r=2.
Then, (u+z+v)+(w+y+u)+(v+x+w)=2.
We also know that u+v+w+x+y+z=1.
We get that u+v+w=1 and x=y=z=0.
Consider the case when the largest side of all dice is on dice A, and that side is rolled.
Then, the only possible outcomes are a>b>c or a>c>b, which have probabilities u and z.
Since z=0, then a>b>c must happen.
Since A is independent of B and C, this means that q, the probability that B beats C, must equal 1.
The cases where B or C have the largest side follow the same logic, by symmetric argument.
Thus, if we assume p+q+r=2, then at least one of p, q, r must equal 1.

Q.E.D

1.4 Property of Four Dice

In this section, we look at what happens to the four dice situation when the sum of probabilities is at the upper limit.

Let A, B, C, D be four dice, and p_1 is the probability that A beats B, p_2 is the probability that B beats C, p_3 is the probability that C beats D, and p_4 is the probability that D beats A. Because $1 \leqslant p_1+p_2+p_3 \leqslant 2$ and $0 \leqslant p_4 \leqslant 1$, then $1 \leqslant p_1+p_2+p_3+p_4 \leqslant 3$.



Proposition: If $p_1+p_2+p_3+p_4$=3, then at least two of p_1, p_2, p_3, and p_4 must equal 1.

Proof:

Let q be the probability that C beats A. Then 1-q is the probability that A beats C.
Then A, B, C and A, C, D are two cycles, and each cycle has three dice.
We know that $p_1+p_2+p_3+p_4$+1-q+q=4, $1 \leqslant p_1+p_2$+q$\leqslant 2$, and $1 \leqslant p_3+p_4$+1-q$\leqslant 2$.
Then p_1+p_2+q=2 and p_3+p_4+1-q=2.
Case 1:
Suppose q equals 1, then 1-q equals 0.
Because p_3+p_4+1-q=2, then p_3+p_4=2.
Since probability should be between 0 and 1, p_3=p_4=1.
Because p_1+p_2+q=2 and q=1, then p_1+p_2=1.
So it is possible that one of p_1, p_2 is 1 and the other is 0, or both p_1 and p_2 are between 0 and 1.
Thus, no matter whether either p_1 or p_2 is 1, p_3 and p_4 equal 1. So at least two of p_1, p_2, p_3, and p_4 must equal 1.
Case 2:
Suppose q is between 0 and 1, then 1-q also is between 0 and 1.
Because p_1+p_2+q=2 and p_3+p_4+1-q=2, one of p_1 and p_2 and one of p_3 and p_4 are 1.
Thus, two of p_1, p_2, p_3, and p_4 must equal 1.
Therefore, if $p_1+p_2+p_3+p_4$=3, then at least two of p_1, p_2, p_3, and p_4 must equal 1.

Q.E.D

So we have shown that when the sum of probabilities is at the upper bound, if we have three dice, then at least one of probabilities must equal 1, and if we have four dice, then at least two of probabilities must equal 1. We infer that if we have n dice and when the sum of probabilities is n-1, then at least n-2 of probabilities must equal 1.

1.5 Property of N Dice

In this section we use induction to prove our statement about n dice.

Proposition: For every number n dice (n⩾3), if we arrange all dice D_1, D_2, D_3, …D_n in a circle and assign p_1, p_2, …p_n be the probabilities that D_1 beats D_2, D_2 beats D_3, …D_n beats D_1, then 1⩽p_1+p_2+p_3+p_4+…+p_n⩽n-1. When the sum equals n-1, at least n-2 of p_1, p_2, …p_n must be 1.

Proof:

Base case:

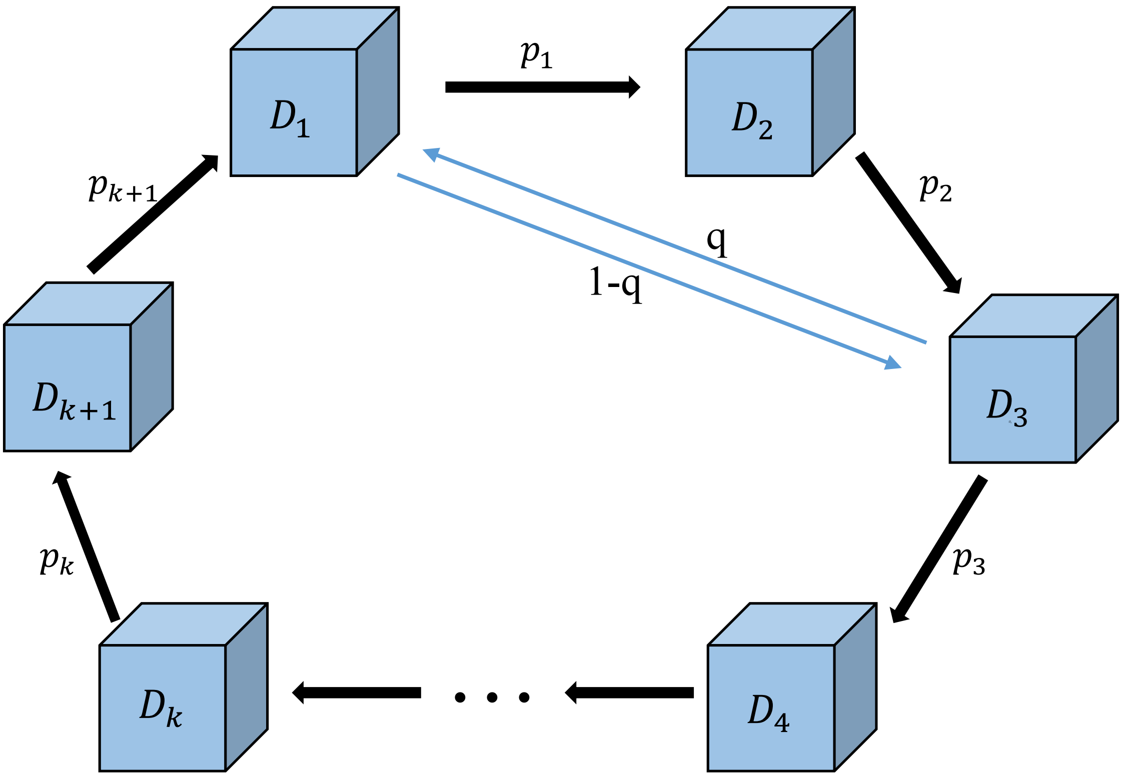
If n=3, we have shown in Section 1.3 that when p_1+p_2+p_3=2, at least n-2=1 of p_1, p_2, p_3 must be 1.

Induction step:

Suppose there are k dice, 1⩽p_1+p_2+…+p_k⩽k-1, and when the sum equals k-1, at least k-2 of p_1, p_2, …$p_k$equal 1.

Let q be the probability that D_3 beats D_1. Then 1-q is the probability that D_1 beats D_3.

Then we have 2 cycles: one cycle has three dice D_1, D_2, D_3 and the other cycle has k dice D_1, D_2, D_3, …D_{k+1}.



Because 1⩽p_1+p_2+p_3+…+p_k⩽k-1 and 0⩽p_{k+1}⩽1, 1⩽p_1+p_2+p_3+…+p_k+p_{k+1}⩽k.

When p_1+p_2+p_3+p_4+…+p_k+p_{k+1}=k, we have p_1+p_2+p_3+p_4+…+p_k+p_{k+1}+(1-q)+q=k+1.

We know 1⩽p_1+p_2+q⩽2, and because of k dice, 1⩽p_3+p_4+…+p_k+p_{k+1}+(1-q)⩽k-1.

Then p_1+p_2+q=2 and p_3+p_4+…+p_k+p_{k+1}+(1-q)=k-1.

Case 1:

Suppose q equals 1, then 1-q equals 0.

Because p_3+p_4+…+p_k+p_{k+1}+1-q=k-1, then p_3+p_4+…+p_k+p_{k+1}=k-1.

Since probability should be between 0 and 1, p_i=1 for i=3, 4, …k+1.

Because p_1+p_2+q=2 and q=1, then p_1+p_2=1.

So it is possible that (1) one of p_1, p_2 is 1 and the other is 0, or (2) both p_1 and p_2 are between 0 and 1.

Thus, no matter whether either p_1 or p_2 is 1, p_3 through p_{k+1} all equal 1, which are k-1 values. So at least (k+1)-2 of p_1, p_2, …p_{k+1} equal 1.

Case 2:

Suppose q is between 0 and 1, then 1-q also is between 0 and 1.

Because p_1+p_2+q=2 and p_3+p_4+…+p_k+p_{k+1} +(1-q)=k-1, one of p_1 and p_2 is 1, and k-2 of p_i is 1 (i=3, 4, …k+1).

Thus, (k+1)-2 of p_1, p_2, …p_{k+1} equal 1.

Case 3:

Suppose q equals 0, then 1-q equals 1.

Because p_1+p_2+q=2, then both p_1 and p_2 are 1.

Because p_3+p_4+…+p_k+p_{k+1}+(1-q)=k-1 and at least k-2 of p_i (i=1, 2, …k) should be 1 in the circle of k dice, then k-3 of p_3+p_4+…+p_k+p_{k+1} should be 1.

Thus, at least k-3+2=(k+1)-2 of p_1, p_2, … p_{k+1} equal 1.

So we have shown that whenever our proposition is valid for any integer k⩾3, k+1 also works for our proposition.

Therefore, for every number n dice (n⩾3), 1⩽p_1+p_2+ …+p_n⩽n-1, and when the sum equals n-1, at least n-2 of p_1, p_2, …p_n must be 1.

Q.E.D

Chapter 2

Background and Motivation

2.1 An Introduction

Social choices procedures have since long shaped the foundations of the society that we live in. The need to elect/choose a representative of the people through some legitimate and logical mechanism has since long bewildered us humans. Perhaps, due to this reason, various voting systems/social choice procedures have been formulated and various communities have different methods/procedures to choose the social choice winner. Any logical system is almost inevitably connected to the field of Mathematics and this field also entails an underlying connection to mathematics. However, complex systems like theses are often fundamentally difficult to comprehend. The aim of this research is to thus decompose this vast voting system to a system of dices. It seems intuitive at first to perceive the voting system that we have as a system of dices. After all, dices are all about probabilities and the the voting system/social choice procedure also rests on the field of probability. It thus would make sense to interpret the voting system as a system of dices in some abstract way. But what exactly could be the analogy and the mathematical foundations of it?

2.2 The analogy between dices and voting

In the traditional voting system, each voter gives a ballot and it contains a ranked preference of all the candidates starting from the most favorite from the least favorite and then based on the voting system we choose, the winner is determined after considering the ballots of all these voters. It is worthwhile to mention that there could be various voting systems and a particular ‘profile' (sequence of ballots of all voters) could yield different social choice winners for different voting systems. Here comes the motivation and the analogy for the dice. We could interpret the candidates standing in the procedure as being dices and the set of outcomes that we get when we roll these dices as being the individual preference lists or ballots. When we roll all the dices once, we get one outcome and that outcome is a ballot. Intuitively, if a dice has very high numbers on its faces, then it would always beat most of the other dices and would emerge as a social choice winner. Lastly, in this grand scheme of things, there are various attributes of social choice procedures like ‘Always-a-Winner’, ‘Condorcet Method’ among others. The voting systems that we use may or may not obey these attributes. And the dice scheme of things may behave differently from the normal voting systems for these various social choice procedures.

Chapter 3

Voting Systems to their Dice counterparts

3.1 Voting System's Attributes

Through our research, we focused primarily on five attributes that any voting system that is said to represent the "will of the people" has. These attributes are Pareto, Monotonicity, IIA, Always a winner, and Condorcet winner. The first three listed have more detail in chapter 4, but briefly :

1. **Pareto** means that if everyone likes A more than B, B shouldn't win;
2. **Monotonicity** means that if A beats B, then a re-election occurs and more people rank A higher, A should still beat B;
3. **IIA** means if A beats B, then a re-election occurs and everyone who liked A more than B still like A more than B and vice versa, A should still beat B.
4. **Always a winner** is self explanatory, this voting system always has a winner.
5. **Condorcet winner**: if by using the condorcet voting method, you get the same winner, then the voting system is said to fit the condorcet winner criteria.

Our first challenge was to describe these in terms of dice rolls, and we found these mirrors for these (again the first 3 have more detail in chapter 4):

1. **Pareto for dice** means that all the sides on dice A are greater than that on dice B.
2. **Monotonicity for dice** means that if dice A beats dice B, then increasing the sides on dice A should still cause it to win.
3. **IIA for dice** means that if A beats B, changing the sides on dice C should have no effect on A beating B.
4. **Always a winner for dice** is still self explanatory, there is always a dice that wins
5. **Condorcet winner for dice** means that you get the same winner using the condorcet method for dice (talked about in the next section)

3.2 Voting Systems

For our research, we focused on six voting systems :

1. **Condorcet** : The Condorcet System is applied by taking one on ones with all the candidates, the winner is the candidate that wins every single one on one. For example, in a system with three candidates, A, B, and C, if A > B and A > C then A wins. However, if we have A > B and A < C, then A does not win here. Thus, in the condorcet system it is possible to not have a winner. Also previously mentioned, but if any of the following voting systems have the same winner when this method is applied to its ballots, then it is said to have the condorcet attribute. Later on we started looking at Copeland's method, which is equivalent to Condorcet's method, but with the difference that if there is no Condorcet winner, then the winner is the Plurality winner (the one with the most first place votes)
2. **Plurality** : As mentioned previously, the Plurality winner is the one with the most first place votes.
3. **Borda Count** : The Borda Count method is looking at where each candidate is placed on each ballot, assigning points based on the position, then summing all points together. The one with the most points is the winner. For example, if my ballot were to be A B C D, then 3 points would go to A, 2 to B, 1 to C, and 0 to D. Then, we apply this to every single ballot, add the points, and the candidate with the most points wins.
4. **Hare's Method** : This is done by looking at which candidate has the least first place votes, removing them, then adjusting the ballots by moving all remaining candidates up if needed, then repeat until one candidate is left.
5. **Dictatorship** : Select one ballot. That ballot decides the winner.
6. **Sequential Pairwise** : This is done by creating an order that you want the candidates to face off in, then the first two go one on one, and the winner of that goes one on one with the third one, and so on. For example, if we decide our order is A, C, D, B then first we do A vs C, and the winner of that goes against D, and the winner of that goes against B.

3.3 Voting Systems and their Attributes

So now that we have desirable traits, and voting systems to focus on, we have to know which voting system has what attributes. The proofs for these are in the textbook *Mathematics and*

Politics : Strategy, Voting, Power, and Proof, but the table is as follows - X means that the system does not have this attribute, Y means it does (note: this is for the voting systems themselves, not for their dice counterparts):

	Pareto	IIA	Monotonicity	Always a Winner	Condorcet winner attribute
Condorcet voting system	Y	Y	Y	X	Y
Plurality	Y	X	Y	Y	X
Borda	Y	X	Y	Y	X
Hare's	Y	X	X	Y	X
Dictatorship	Y	Y	Y	Y	X
Seq Pairs	X	X	Y	Y	Y

As we can see, no voting system has every single positive attribute. In fact, we have that the only system that has Monotonicity, IIA, and Pareto (arguably the 3 most important) is a dictatorship - This is Arrow's Theorem (which will be covered more in chapter 4)

3.4 Converting Dice to Voting Systems

One goal of our research was to convert voting systems in vote theory into equivalent systems with dice. We accomplished this by taking the probability of each outcome, then assigning an equivalent amount of voters to it - by creating a common denominator for all probabilities, then in our system the number of voters would be the denominator, and each individual ballot would be the numerator of that probability. For example, if our outcomes were A > B > C with probability $\frac{1}{6}$, B > A >C probability $\frac{1}{3}$, and C > B > A probability $\frac{1}{2}$, then our voting system would have 1 ballot for ABC, 2 ballots for BAC, and 3 ballots for CBA, for a total of 3 ballots. After applying the method like this, we would proceed with the standard definition of each voting system. The only major difference is with dictatorship, where we decided that dictatorship for dice would be that we only focus on one outcome.

3.5 Dice Systems and Their Attributes

Looking at our conversions from voting systems to their dice system, we wanted to see that they would have all their attributes. Our hypothesis is that the dice systems would maintain all the attributes, and while we do not have every single attribute, we have most and what we have so far is that they do line up. The basic idea behind proving these is either by finding an example, or using algebra/logic to show that they do in fact have the quality.

	Pareto	IIA	Mono	Always a Winner	Condorcet winner attribute
Condorcet Dice System	Y	Y	X	?	?
Plurality	Y	X	Y	Y	X
Borda	Y	X	Y	Y	X
Hare’s	Y	X	X	Y	?
Dictatorship	Y	Y	Y	Y	X
Seq Pairs	X	X	Y	Y	Y

We believe that the reason they line up is that we map the create a function F : A -> B where A is the dice rolls, and B is the voting system, then F is injective (but not surjective - why we believe Arrow’s Theorem failed) and thus we have F*: B -> A, which allows the properties of the voting systems to be maintained.

Chapter 4

Proving Arrow's Theorem for Dice

4.1 Limitation in Mapping Dice to Voters

When we first began our process of mapping dice to voters, we ran into problems quickly because of the restraint that dice outcomes have on voters.

> **Crucial Observation:** In voting theory, all possible outcomes of voters exist. With dice, we can only pick the labelling of the dice sides, and this restricts our set of outcomes.

This was an issue we tried to work around when we could not exactly find a one-to-one correspondence with our dice analogy for monotonicity, but ended up being a big factor in our findings when we worked with Arrow's Theorem. Labelling the dice sides gives us a set of outcomes, but we cannot change a single voter like we could in voting theory. The sides define our set of voters, and it is difficult (often impossible) to get the exact outcomes we need in some situations when we want to observe phenomena that simply cannot exist when we are restrained to only voters given by dice outcomes.

4.2 Dice Analogies for Voting System Attributes

When we first began working on proving Arrow's Theorem in our subset of voters limited by dice, we first had to define what voting system attributes mean for dice. We have already previously defined what voting systems look like for dice, so we can now use the three attributes for voting systems to observe and quantify what properties a "fair" voting system has. The attributes we chose to observe are the Pareto condition, Monotonicity, and Independence of Irrelevant Alternatives (IIA).

4.2.1 Pareto condition

The resource we used, *Mathematics and Politics* by Taylor and Pacelli, defined the Pareto condition (often abbreviated Pareto) for voting systems to be where:

> If everyone prefers x to y, then y is not a social choice.

In this definition, social choice is the winner of the ballots. We observed that this was an extremely weak condition, since it is not common that every voter prefers one candidate over another. That being said, we defined Pareto to be:

> If dice *A* has all sides greater than all sides on dice *B,* dice *B* should not win.

We define a dice to win when it rolls the highest side in any given roll of the dice. We wrote this to be the definition of Pareto for dice because in this case, when we map the dice outcomes to voters, we can see that dice *A* will always be higher than dice *B* in every voter's ballot, which can directly be translated to everyone preferring the candidate *A* to candidate *B.* Following this logic, we can see that if everyone prefers candidate *A* to candidate *B*, then candidate (dice) *B* should not win.

4.2.2 Monotonicity

In voting theory, monotonicity is defined to hold for every candidate *x*:

> If *x* is the social choice and someone changes their preference list to move *x* up one spot, then *x* should still be the social choice (Taylor, Pacelli).

This means that when we theoretically improve the winner's chances, then that candidate should still win. When we defined monotonicity for dice, we decided that we can change the side on a dice to be higher to increase that dice's chances of winning. This leads us to our definition of monotonicity for dice:

> If dice *A* beats dice *B*, and we change one side of dice *A* that was less than than a side on *B* to be larger than that side, dice *A* should still beat *B*.

This naturally follows from our definition of increasing a dice/candidate's chances in the election. There are a few issues with this definition because of the fact that in voting theory, we only move the candidate *x* up in *one* ballot, while in dice, changing one side on one dice may change some or all of the ballots. This is a big difference, since we cannot just change one side of dice without changing multiple outcomes. The definition of monotonicity also states that the voter only moves the candidate up one spot, whereas in our definition, it may be that changing one side might move that candidate up many spots in the ballots depending on what the changed side becomes.

This means that our definition of monotonicity is a bit different from what it means for a voting system to be monotone in voting theory, but we tried to work with this definition throughout the quarter anyway.

4.2.3 Independence of Irrelevant Alternatives (IIA)

IIA is defined in voting theory to be:

> For every pair of alternatives *x* and *y*, if *x* beats *y*, and one or more voters change their ballots but do not change whether they like *x* over *y* or *y* over *x,* then *x* should still beat *y* (Taylor, Pacelli).

This means that between any two candidates *x* and *y*, as long as no one changes their mind about their preferences on these two candidates, then between *y* should still not win. Since we wanted their positions to remain the same relative to each other in every single ballot, we define IIA in dice as:

> If Dice *A* beats dice *B*, then changing the sides on dice *C* should not impact dice *A* beating dice *B*.

This gives us the freedom to change all of the sides on any dice besides dice *A* and dice *B.* Since we will not change the sides of *A* and *B,* we know that the relative outcomes within each ballot/outcome between each other will remain the same.

4.3 Limitations in Proving Arrow's Theorem for Dice

In voting theory, Arrow's Theorem is defined as

> If a voting system satisfies both Pareto and IIA, then that voting system is a dictatorship.

Naturally, since we had defined dice analogues for Pareto and IIA, we wanted to prove that this theorem would hold in the subset of voters represented by dice outcomes. That is, we wanted to prove Arrow's Theorem using our dice analogues for Pareto, IIA, and the voting systems which we had defined.

We began trying to prove Arrow's Theorem for dice using the lemmas that were used to prove Arrow's Theorem in voting systems in *Mathematics and Politics* (Taylor, Pacelli). Beginning with the first lemma, we had already ran into an issue, going back to the crucial observation in 4.1. We are very restrained when we limit our set of voters to the subset of ballots we receive from dice outcomes, and this prevented us from proving the first lemma of the proof we used when we tried to prove Arrow's Theorem.

The first lemma in the proof of Arrow's Theorem gave us three subsets of voters, which we had to map to our dice outcome voters. These three subsets made up the entire set of voters, and listed their preferences for three different candidates. In voting theory, we can define these voters' ballots to be the preferences we want them to be, which is why this lemma can be proven easily in voting theory. In dice however, we could not define what it means for three outcomes to be the only three possible outcomes for all of the dice. This goes back to our theorem proven in chapter 1, where we added three dice probabilities and observed the equality on the upper bound. We proved that if three dice probabilities added up to the upper bound, 2, then one of the probabilities must be zero. This means that when we try to define these subsets of voters as dice outcomes, then one of the sets *must* be empty. This is a case that might work/be covered by the lemma, but does not help us prove Arrow’s Theorem because we cannot prove the lemma for all sets of voters where this occurs, since we are already given the sets of voters.

Therefore, we cannot prove Arrow’s Theorem (at least, using the lemmas that our resource used to prove Arrow’s Theorem) for dice, as we are too restrained by the subset of voters given by dice outcomes.