WXML Final Report: Number Theory and Noise

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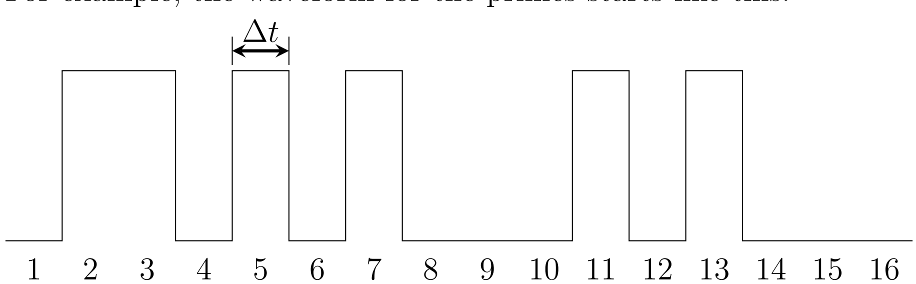
1 Introduction

Student in this project investigate the possibilities arising from representing sets of positive integers as sound.

A digital audio file is created from a given set A of positive integers by setting sample number i to a non-zero constant c for all i in the set. All other samples are set to zero.

For example, the waveform for the primes starts like this:



We use the standard CD-audio sampling rate of 44100 samples per second, so $\Delta t = \frac{1}{44100} = 0.0000226757...$ seconds.

For many sets, the result is what most people would describe as *noise*.

Students worked on this project last Spring, and new students continued the project this quarter. We started with four students, but two dropped out. The remaining two, Hannah Van Wyk and Jesse Rivera, were very productive and we all had many fascinating discussions.

This quarter, the focus was on sequences in the Online Encyclopedia of Integer Sequences (OEIS). The encyclopedia provided a huge variety of sequences to experiment with, with clear definitions and (at least some) context.

Sound files for more than 50 integer sequences were created this quarter. They can be heard here:

`http://www.math.washington.edu/~conroy/sequenceNoise/indexWXMLaut2016.htm`

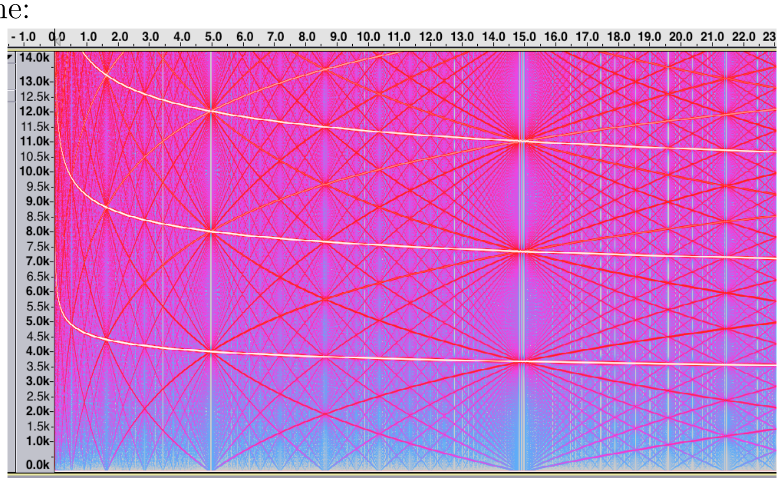
Commentary on some of Hannah's and Jesse's work is below.

2 Progress

2.1 Hannah Van Wyk's commentary

Out of all the different types of number sequences we explored throughout the quarter, the ones that sounded the most unpredictable and enigmatic to me were the Beatty sequences. Beatty sequences are advantageous because they allow us to use functions which would normally be impractical in our study of integer sequences (such as logarithmic functions or trigonometric functions). The Beatty sequences include only the integer value of an irrational number in a sequence by rounding down the number (also known as the floor function).

The sound for $\lfloor n \log n \rfloor$ stands out as particularly unusual. The pitches in the sequence are very clearly defined with no gritty effects. These pitches are decreasing or increasing in frequency very smoothly (at least up until 23 seconds) much like a few staggered police sirens would sound in slow motion. Here is a spectrogram of the sound which shows frequency plotted against time:



My group discovered a characteristic of the sequence which may explain the unusual nature of the sound and the spectrogram. When you make a new sequence by subtracting every nth term of the sequence from the $n+1$st term to create the sequence of consecutive differences of $\lfloor n \log n \rfloor$, you get this:

$$1, 2, 2, 3, 2, 3, 3, 3, 4, 3, 3, 4, 3, 4, 4, 4, 4, 3, 4, 4, 5, 4, 4, 4, 4, 4, 5, 4, 5,$$

$$4, 4, 5, 4, 5, 5, 4, 5, 4, 5, 5, 4, 5, \ldots$$

This shows a few interesting things. The first is that the differences increase by no more than one at a time. It also appears to take longer for the differences to increase as they get higher (there are only 3 2s before they stop reappearing but there are 8 3s.) This has something to do with the characteristics of a Beatty sequence and the logarithmic function. We are still unsure about why exactly this sequence of differences of $\lfloor n \log n \rfloor$ creates the sound it does, but we are hoping to continue exploring this in the future.

2.2 Jesse Rivera's commentary

2.2.1 Partial sums of solutions to the Josephus problem (A256249):

This sequence stood out to me because of its similarities to another simpler sequence; it sounds exactly like the sequence of squares, only with multiple beginnings. The sequence of squares has gaps of $1, 3, 5, 7, \ldots$. This sequence has gaps of $1, 1, 3, 1, 3, 5, 7, 1, 3, 5, \ldots$. So the gaps in this sequence follow the same pattern, only in this sequence the gaps reset to 1 every time the next highest power of 2 is reached. This results in the two sequences sounding identical, only in the second sequence the sound repeatedly restarts after a certain period of time, with this period of time increasing after each repetition.

2.2.2 Numbers in base 5 (A007091):

This sequence is interesting in that its sound has a fractal waveform. There is a pattern of five distinct sounds grouped together, with each distinct sound separated by a period of silence. There are then five of these groups separated by periods of silence that are a magnitude of 10 larger than the previous smaller gaps, and so on. The smallest subgroup consists of the numbers $0-4$, $10-14$, $20-24$, $30-34$, $40-44$, which are separated by gaps of 10.

The next group is the subgroups $0-44$, $100-144$, $200-244$, $300-344$, $400-444$, which are separated by gaps of 100. This pattern continues as the sequence progresses.

2.2.3 Trigonometric sequences, e.g. positive integers n such that $\sin n < 0$ and $\sin(n+2) < 0$ (A277096):

These sequences produced sounds containing multiple clear and steady tones. Spectrograms reveal dominant frequencies that indicate patterns occurring at multiples of 2π in the sequence, however, there cannot be patterns occurring regularly at multiples of 2π since the sequence is composed entirely of integers. Furthermore, differences between consecutive terms of the sequence are constrained to 1, 6, and 7. We are still unsure of what is causing these frequencies, although it is intuitive for pi to show up in sequences defined by trigonometric functions. I suspected that the average of the gaps might be 2π, but it is not (it turns out to be approximately 5.5 for the first $150,000$ terms of the sequence).

2.2.4 Connell Sequences (A001614, A045928, A033291, etc.):

I found the Connell sequences to be interesting because it is possible to alter the definition of the standard sequence and end up with similar yet unique sounds. The Connell sequence is defined as follows: take the first odd number, then the next two even numbers, then the next three odd numbers, etc. This can be thought of a sequence of subsequences, with the first subsequence being 1, the next subsequence being 2, 4, the next being 5, 7, 9, and so on. The standard Connell sequence sounds exactly like the squares; this is because the sequence is composed entirely of gaps of 2, with gaps of 1 occurring after each square. The fact that the gap is different at each square is what causes the sequence to sound identical to the squares. There exists, however, an ultrasonic tone (with a frequency of $22,050$ Hz) caused by the gaps of 2. Since we are unable to hear this tone, the sequence sounds exactly like the sequence of squares. I also thought that the standard Connell sequence was interesting in that the sequence of partial sums of this sequence sounds strikingly similar to the sequence itself.

The Connell sequence can be generalized to having two parameters: the first being differences in consecutive terms of each subsequence, and the second being how much longer each subsequence is than the previous. By this definition the standard Connell sequence would have parameters of $2, 1$. Increasing the first parameter lowers the frequency of the steady underlying tone of the sound, while increasing the second parameter causes the sound to resemble higher figurate sequences.

Generalizing the Connell sequence even further, one can define sequences such as A033291 (take the first multiple of 1, the next two multiples of 2, the next three multiples of 3, etc.). The resulting sequences sound similar in the sense that they all resemble the figurate sounds (beginning with a high pitched tone that gradually decreases), yet each sequence has its own unique sound and is easily distinguishable from other Connell-like sequences.

2.2.5 Number of odds in the first n rows of Pascals triangle (A006046):

I found this sound to be interesting because it represents a characteristic of Pascals triangle, a mathematical structure with many notable features. Gaps in the sequence signify how many odds are in a given row (i.e. a long duration of silence indicates many odds). This sequence is strictly increasing since each row of Pascals triangle begins and ends with 1, an odd number. The sound has a repeating pattern that occurs regularly, and even far into the sequence there are both small and large gaps, signifying that there are rows deep in Pascals triangle that consist of mostly odds, followed by rows containing very few odds.

3 Future directions

The OEIS is vast, so there is really no end in sight for this general work of creating sounds. This quarter, though, it has become clear that some exploration of Fourier analysis (and signal analysis generally) would be helpfull for addressing many questions that have arisen from the work.

Beatty sequences in particular will be a great starting point for exploring Fourier analysis of these sounds, as they are just complex enough to exhibit surprising phenomena. I anticipate at least some future project members spending a good deal of effort on these sequences.