Towards Fast Computation of Exit Probabilities for Maxima of Randomized Brownian Bridges

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Summary

Motivated by a financial application, we explore $\mathbb{P}(M^* \geq K)$, where M^* is the maximum of a randomized brownian bridge on $[0, t]$ from $a \in \mathbb{R}$ to E, E being some continuous random variable with density.

Introduction

The probability that the maximum price that a stock could take between now and the end of the trading period exceeds a given value K is of interest to those trading (American) options for this stock. Suppose a trader knew that the probability that the maximum future stock price (given the current price) exceeds P_s, the strike price, plus P_o, the current option price, were greater than .5. Then it is in the trader's interest to buy options (exactly how many is left to the trader's discretion, although the larger the aforementioned probability, the more options one would consider buying), as it is more likely than not that the maximum future stock price will be some number $P_{max} > P_s + P_o$, in which case the trader could exercise the purchased options, sell the corresponding stocks, and obtain a profit [1] of $N\left(P_{max} - (P_s + P_o)\right)$, N being the number of options originally purchased. Therefore, if the future stock price is modeled as a randomized brownian bridge on the interval $[0, t], t > 0$ (t represents the length of the remaining trading period) from P_{curr}, the current stock price, to E, a random variable with density (representing the trader's best guess of the distribution of stock prices at the end of the trading period), letting M^* denote the maximum possible value of this bridge, fast computation of $\mathbb{P}(M^* \geq K)$ for arbitrary K (one might want to compute the probability that $M^* \geq P_s + P_o$ for many different values of P_s and P_o, for instance, if one were precomputing these probabilities in advance) is of interest.

Progress

We first proved the following lemma:

[1] Roughly speaking. The tax can be factored into P_s.

Lemma 1. *Let $a, b \in \mathbb{R}, t > 0$. Let M^+ be the maximum of a brownian bridge from a to b on $[0, t]$. Then*

$$\mathbb{P}\left(M^+ \geq y\right) = \begin{cases} e^{\frac{-2ab}{t}} e^{\frac{-2y^2}{t} + \frac{2ay}{t} + \frac{2by}{t}} & \text{if } y > \max(a, b) \\ 1 & \text{if } y \leq \max(a, b) \end{cases} \tag{1}$$

The proof follows from formula 1.1.8 on p160 in [BS15], reproduced below:

$$\mathbb{P}_a\left(\sup_{0\leq s\leq t} W_s \geq y, W_t \in dz\right) = \frac{1}{\sqrt{2\pi t}} e^{-(|z-y|+y-a)^2} dz \qquad (a \leq y) \tag{2}$$

See the appendix for a complete proof. We then proved the following theorem:

Theorem 1. *Let $a \in \mathbb{R}, t > 0$. Let M^* be the maximum of a brownian bridge from a to E on $[0, t]$, where E is a continuous random variable having density function f_E. Let f_{M^*} denote the density function of M^* and $f_{M^+}(*; b)$ denote the density function of the maximum of a brownian bridge from a to b. Assume that there exists a function $B : \mathbb{R} \to \mathbb{R}$ such that $\int_K^\infty B(x)dx$ converges and that for all but possibly finitely many x,*

$$\left| f_{M^*}(x) - \int_{-\infty}^{\infty} f_E(b) f_{M^+}(x; b) db \right| \leq B(x) \tag{3}$$

Then:

$$\left| \mathbb{P}(M^* \geq K) - \left[\mathbb{P}(E \geq K) + \int_{-\infty}^{K} f_E(b) e^{\frac{-2ab}{t} - \frac{2K^2}{t} + \frac{2a}{t}K + \frac{2b}{t}K} db \right] \right| \leq \int_K^\infty B(x)dx \tag{4}$$

Proof. Noting that:

$$\begin{aligned} \{b \in \mathbb{R} : K \geq \max(a, b)\} = (-\infty, K] \\ \{b \in \mathbb{R} : K < \max(a, b)\} = (K, \infty) \end{aligned} \tag{5}$$

that by the lemma, for any $b \in \mathbb{R}$,

$$\int_K^\infty f_{M^+}(x; b)dx = \begin{cases} e^{\frac{-2ab}{t} - \frac{2K^2}{t} + \frac{2aK}{t} + \frac{2bK}{t}} & \text{if } K > \max(a, b) \\ 1 & \text{if } K \leq \max(a, b) \end{cases}$$

we have that:

$$-B(x) \leq f_{M^*}(x) - \int_{-\infty}^{\infty} f_E(b) f_{M^+}(x; b) db \leq B(x) \tag{6}$$

$$-\int_K^\infty B(x)dx \leq \int_K^\infty f_{M^*}(x)dx - \int_K^\infty \int_{-\infty}^{\infty} f_E(b) f_{M^+}(x; b) db dx \leq \int_K^\infty B(x)dx \tag{7}$$

$$\left|\mathbb{P}(M^* \geq K) - \int_K^\infty \int_{-\infty}^\infty f_E(b) f_{M^+}(x;b) db dx\right| \leq \int_K^\infty B(x)dx \tag{8}$$

However, by Fubini's theorem,

$$\int_K^\infty \int_{-\infty}^\infty f_E(b) f_{M^+}(x;b) db\ dx = \int_{-\infty}^\infty f_E(b) \int_K^\infty f_{M^+}(x;b) dx db \tag{9}$$

and,

$$\int_{-\infty}^\infty f_E(b) \int_K^\infty f_{M^+}(x;b) dx db \tag{10}$$

$$= \int_{\{b \in \mathbb{R} : K > \max(a,b)\}} f_E(b) \int_K^\infty f_{M^+}(x;b) dx db$$

$$+ \int_{\{b \in \mathbb{R} : K \leq \max(a,b)\}} f_E(b) \int_K^\infty f_{M^+}(x;b) dx\ db$$

$$= \int_{-\infty}^K f_E(b) e^{\frac{-2ab}{t} - \frac{2K^2}{t} + \frac{2aK}{t} + \frac{2bK}{t}} db + \int_K^\infty f_E(b) db$$

$$= \mathbb{P}(E \geq K) + \int_{-\infty}^K f_E(b) e^{\frac{-2ab}{t} - \frac{2K^2}{t} + \frac{2aK}{t} + \frac{2bK}{t}} db$$

□

Remark. *It is believed that in many cases of interest, the function $B = 0$ may be used to satisfy the hypotheses of the theorem. See Figures 1 and 2 for numerical evidence of this. However, it is not known in general whether:*

$$f_{M^*}(x) = \int_{-\infty}^\infty f_E(b) f_{M^+}(x;b) db \qquad \textit{for all but possibly finitely many } x \tag{11}$$

Our previous attempts at proving the above equation have been unsuccessful as it appears that E and M_E, the maximum of a brownian bridge from a to E, need not be jointly continuous; referring to Definition 13.3 in [JP], given that $\mathbb{P}[E \in (-\infty, a)] \neq 0$, we believe that, at least in some cases, $N = (-\infty, a) \times \{a\}$ is a counterexample against the claim that E and M_E are jointly continuous; see Figure 3 for numerical evidence of this.

Conclusion

The theorem could potentially be used for fast computation of $\mathbb{P}(M^* \geq K)$. For example, in the case that $E \sim \mathcal{N}(\mu, \sigma^2)$ and assuming that the hypotheses of the theorem can be satisfied with $B = 0$, we have that for any $K > a$,

$$\mathbb{P}(M^* \geq K) = \mathbb{P}(E \geq K) + \frac{\left(-\Delta erf\left(\frac{|\Delta|}{\sqrt{2t\sigma}}\right) + |\Delta|\right) exp\left(\frac{2(a-K)(a\sigma^2 + K(t-\sigma^2) - \mu t)}{t^2}\right)}{2|\Delta|} \tag{12}$$

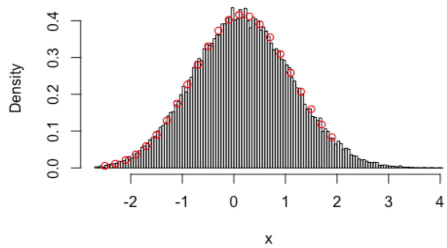

Figure 1: $a = -2.7, E \sim \mathcal{N}(0, 1)$. Selected values of the right hand side of Equation 11 (a closed form for $f_{M^+}(x, b)$ was obtained and used in computations) plotted in red over the empirical density of M^* (50000 simulations).

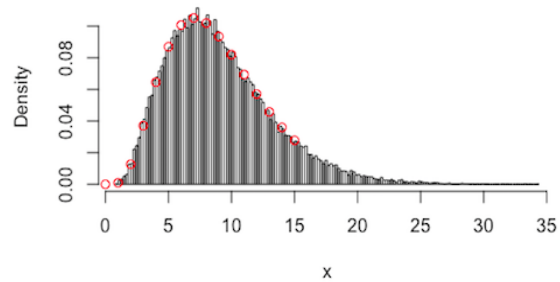

Figure 2: $a = 0, E \sim \mathcal{X}^2(9)$. Selected values of the right hand side of Equation 11 (a closed form for $f_{M^+}(x, b)$ was obtained and used in computations) plotted in red over the empirical density of M^* (50000 simulations).

nsim	10 000	20 000	30 000	40 000	50 000	60 000	70 000	80 000	90 000	100 000
Prob.	0.3902	0.38835	0.390567	0.389	0.393	0.389533	0.386829	0.390363	0.3926	0.38778

Figure 3: Simulated values (represented up to at most 6 decimal places) for $\mathbb{P}\left[(E, M_E) \in (-\infty, 0) \times \{0\}\right]$ when $E \sim \mathcal{N}(-15, 1), a = 0$. For each value of nsim, that many values of (E, M_E) were generated. Note that this probability would be zero if E and M_E were jointly continuous.

where $\Delta = -2a\sigma^2 + 2K\sigma^2 - Kt + \mu t$. Thus, given reasonably fast implementations of $normCDF$ and $erf(x)$, fast computation of $\mathbb{P}(M^* \geq K)$ should follow. Further work is needed to examine Equation 11 to potentially refine the hypotheses of the theorem. Further work is also needed to compare the performance of the consequences of the theorem (such as the above equation) against other methods for computing $\mathbb{P}(M^* \geq K)$.

This project has many possible future directions. All brownian bridges previously mentioned in this paper have unit variance (for an introduction to brownian bridges, see [Sie]), yet one might want to compute $\mathbb{P}(M^* \geq K)$ in the context of a stock with internal variance other than one. Furthermore, we have not yet attempted to obtain a theorem similar to Theorem 1 for discrete, rather than continuous, endpoints E.

Finally, so far we have assumed that a brownian bridge with randomized right endpoint can be used to used to model future stock prices. However, a brownian bridge can be negative, yet a stock price can never be negative. Further work is needed to examine how the frequency at which a brownian bridge with randomized right endpoint dips below the x-axis depends on, say, the start of the bridge and the distribution of the right endpoint (one could potentially prove a theorem similar to Theorem 1 concerning the minimum of a randomized brownian bridge, rather than the maximum), and to find workarounds if this frequency is too large in cases of interest. Geometric brownian motion might be useful in this regard.

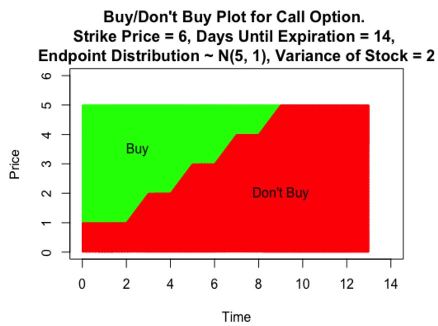

Figure 4: 6-point discretizations were used to approximate the brownian bridges. Buy if $\mathbb{P}(\text{Max} \geq \text{strike}) > .5$, don't buy otherwise.

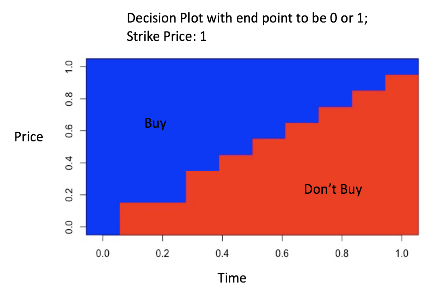

Figure 5: The endpoint equals 0 with probability $1/2$. Buy if $\mathbb{P}(\text{Max} \geq \text{strike}) > .5$, don't buy otherwise.

Bibliography

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[JP] Krishna Jagannathan and Subrahmanya Swamy P. *Lecture 13: Conditional Distributions and Joint Continuity*. URL: `http://nptel.ac.in/courses/108106083/lecture13_Conditional_Distributions_and_Joint_Continuity.pdf`.

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Appendix

The lemma is proven below.

Proof. Let (B_t) be a brownian bridge on [0, t] from a to b,

$$M^+ := \sup_{0\le s\le t} B_t.$$

It is well known (see p10 and p54 of [MP08], for instance) that if W_t is a 1-dimensional brownian motion started at a, then for any $t > 0$,

$$\mathbb{P}(W_t \in S) = \int_S \mathfrak{p}(t,a,z)dz, \quad \text{where} \quad \mathfrak{p}(t,a,z) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{(a-z)^2}{2t}}$$

Suppose that $y \ge a$ is arbitrary. Then

$$\mathbb{P}(M^+ \ge y) = \mathbb{P}\left(\sup_{0\le s\le t} B_t \ge y\right) \tag{13}$$

$$= \lim_{\epsilon\to 0} \mathbb{P}_a(\sup_{0\le s\le t} W_t \ge y \mid |W_t - b| \le \epsilon) \qquad (W_t \text{ is a brownian motion on [0, t] starting at } a) \tag{14}$$

$$= \lim_{\epsilon\to 0} \frac{\mathbb{P}_a(\sup_{0\le s\le t} W_t \ge y, |W_t - b| \le \epsilon)}{\mathbb{P}_a(|W_t - b| \le \epsilon)} \tag{15}$$

$$= \lim_{\epsilon\to 0} \frac{\mathbb{P}_a(\sup_{0\le s\le t} W_t \ge y, |W_t - b| \le \epsilon)}{\mathbb{P}_a(b-\epsilon \le W_t \le b+\epsilon)} \tag{16}$$

$$= \lim_{\epsilon\to 0} \frac{\int_{b-\epsilon}^{b+\epsilon} \frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}dz}{\int_{b-\epsilon}^{b+\epsilon} \frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}dz} \qquad \text{(For the numerator, see [BS15], formula 1.1.8 on p160)} \tag{17}$$

$$= \lim_{\epsilon\to 0} \frac{\frac{d}{d\epsilon}\int_{b-\epsilon}^{b+\epsilon} \frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}dz}{\frac{d}{d\epsilon}\int_{b-\epsilon}^{b+\epsilon} \frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}dz} \qquad \text{(by L'Hospital's rule)} \tag{18}$$

$$= \lim_{\epsilon\to 0} \frac{\frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}\Big|_{b+\epsilon}\frac{d}{d\epsilon}[b+\epsilon] - \frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}\Big|_{b-\epsilon}\frac{d}{d\epsilon}[b-\epsilon]}{\frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}\Big|_{b+\epsilon}\frac{d}{d\epsilon}[b+\epsilon] - \frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}\Big|_{b-\epsilon}\frac{d}{d\epsilon}[b-\epsilon]} \tag{19}$$

$$= \lim_{\epsilon\to 0} \frac{\frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}\Big|_{b+\epsilon} + \frac{1}{\sqrt{2\pi t}}e^{-(|z-y|+y-a)^2/(2t)}\Big|_{b-\epsilon}}{\frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}\Big|_{b+\epsilon} + \frac{1}{\sqrt{2\pi t}}e^{-(a-z)^2/(2t)}\Big|_{b-\epsilon}} \tag{20}$$

$$= \lim_{\epsilon \to 0} \frac{\frac{1}{\sqrt{2\pi t}} e^{-(|b+\epsilon-y|+y-a)^2/(2t)} + \frac{1}{\sqrt{2\pi t}} e^{-(|b-\epsilon-y|+y-a)^2/(2t)}}{\frac{1}{\sqrt{2\pi t}} e^{-(a-(b+\epsilon))^2/(2t)} + \frac{1}{\sqrt{2\pi t}} e^{-(a-(b-\epsilon))^2/(2t)}} \tag{21}$$

$$= \frac{\frac{2}{\sqrt{2\pi t}} e^{-(|b-y|+y-a)^2/(2t)}}{\frac{2}{\sqrt{2\pi t}} e^{-(a-b)^2/(2t)}} \tag{22}$$

$$= e^{\frac{(a-b)^2}{2t}} e^{\frac{-(|b-y|+y-a)^2}{2t}} \tag{23}$$

Therefore, if $y > \max(a, b)$ is arbitrary, $y \geq a$ and $|b - y| = y - b$, so

$$\begin{aligned} \mathbb{P}\left(M^+ \geq y\right) &= e^{\frac{(a-b)^2}{2t}} e^{\frac{-(y-b+y-a)^2}{2t}} \\ &= e^{\frac{-2ab}{t}} e^{\frac{-2y^2}{t} + \frac{2ay}{t} + \frac{2by}{t}} \end{aligned} \tag{24}$$

Furthermore, for any $y \leq \max(a, b)$, $\mathbb{P}(M^+ \geq y) = 1$, as M^+ is the maximum of a brownian bridge starting at a and ending at b. Thus,

$$\mathbb{P}\left(M^+ \geq y\right) = \begin{cases} e^{\frac{-2ab}{t}} e^{\frac{-2y^2}{t} + \frac{2ay}{t} + \frac{2by}{t}} & \text{if } y > \max(a, b) \\ 1 & \text{if } y \leq \max(a, b) \end{cases} \tag{25}$$

□