WXML Final Report: Algebraic Combinatorics

Faculty mentors: Sara Billey and Philippe Nadeau
Graduate mentor: Jordan Weaver
Undergraduates: Jesse Rivera and Justin Shyi

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1 Notation

$\lfloor x \rfloor$ is the floor function, which gives the largest integer less than or equal to $x \in \mathbb{R}$.

$n!! = \prod_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2k) = n \cdot (n-2) \cdot (n-4) \cdots$ is the double factorial.

$w_0 = [n, n-1, \ldots, 1] \in S_n$ is the permutation mapping $x \mapsto n+1-x$.

Unless otherwise specified, we write permutations in one-line notation.

2 Introduction

In this project we investigate a certain type of algebraic structure found in symmetric groups. Our work builds off of the theory developed in the paper "Parabolic Double Cosets in Coxeter Groups," authored by Sara Billey, Matjaž Konvalinka, T. Kyle Petersen, William Slofstra, and Bridget Tenner. We will provide a brief summary of this theory in the following section for the purpose of defining terminology and contextualizing our results. A much more comprehensive treatment of this material can of course be found in [1].

3 Background

We begin with some definitions. An *adjacent transposition* is a permutation that swaps a single pair of adjacent elements. One example of this is $[12435] \in S_5$, which we denote s_3 because it swaps the elements $(3 \leftrightarrow 4)$. In our work it is useful to think of the symmetric group as being generated by the adjacent transpositions, $S_n = \langle s_i \mid i = 1, 2, \ldots, n-1 \rangle$. A *parabolic subgroup* of S_n is any subgroup that can be generated by adjacent transpositions. That is, a parabolic subgroup is a subgroup of the form $W_I = \langle s_i \mid i \in I \rangle$ where $I \subseteq \{1, 2, \ldots, n-1\}$. A *parabolic double coset* is then defined to be a two-sided coset with respect to two parabolic subgroups, $W_I w W_J = \{pwq \mid p \in W_I, q \in W_J\}$.

The underlying question that motivates our work is simple: How many parabolic double cosets are in S_n? Due to the complex structure of parabolic double cosets, counting these objects requires more work than one might expect. This question is answered in [1] in the context of finitely generated Coxeter groups.

The *length* of a permutation $w \in S_n$ is formally defined to be the minimum number k of adjacent transpositions in a reduced expression $s_{i_1} \cdots s_{i_k} = w$. It is equal to the number of inversions of w and is denoted $\ell(w)$. That is, $\ell(w) = \#\{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$. One very useful fact is that every parabolic double coset has a unique minimal length element. We define right and left ascent and descent sets of a permutation $w \in S_n$ as follows:

$$\text{Asc}_R(w) = \{1 \leq j \leq n-1 \mid w(j) < w(j+1)\}$$
$$\text{Des}_R(w) = \{1 \leq j \leq n-1 \mid w(j) > w(j+1)\}$$

and $\text{Asc}_L(w) = \text{Asc}_R(w^{-1})$ and $\text{Des}_L(w) = \text{Des}_R(w^{-1})$. It can be shown that a permutation w is the minimal length element of the parabolic double coset $W_I w W_J$ if and only if $I \subseteq \text{Asc}_L(w)$ and $J \subseteq \text{Asc}_R(w)$.

One should notice that there are multiple ways of writing the same parabolic double coset. For this reason we define a *presentation* of a parabolic double coset $C \subseteq S_n$ to be a triple (I, w, J) such that $C = W_I w W_J$. We say that a presentation is *lex-minimal* if w is the minimal length element of $W_I w W_J$ and if $W_I w W_J = W_{I'} w' W_{J'}$ then either $|I| < |I'|$ or $|I| = |I'|$ and $|J| \leq |J'|$. This is a very important notion to establish if our goal is to count parabolic double cosets. In particular, every lex-minimal presentation corresponds to a unique parabolic double coset, and it is proven in [1] that every parabolic double coset has a unique lex-minimal presentation.

We will use C_w to denote the set of parabolic double cosets whose minimal-length element is w, C_w^* to denote the set of lex-minimal presentations of elements in C_w, and $c_w = |C_w^*| = |C_w|$ to denote the number of parabolic double cosets whose minimal-length element is w. The main formula presented in [1] counts the number of parabolic double cosets in S_n by summing c_w over all permutations $w \in S_n$. One of the main reasons this project exists is because the authors believed there to be a more efficient way of counting parabolic double cosets. Our primary goal is to find such a way or prove that one does not exist. We have made some progress in this regard.

4 Progress

We began our work by collecting some data on S_n for small n. In particular, we looked at the values c_w for each permutation $w \in S_n$. This led us to the question of when $c_w = c_{w'}$ for two permutations $w \neq w'$. We were able to prove the following.

Lemma 1. *Let $g : S_n \to S_n$ be given by $g(w) = w^{-1}$, and let $w \in S_n$ be an arbitrary permutation. Consider the function $G : C_w \to C_{w^{-1}}$ mapping a parabolic double coset to its image under g. Then G is a bijection so that $c_w = c_{w^{-1}}$.*

Proof. We will first show that for any parabolic double coset $W_I w W_J \subseteq S_n$, $g(W_I w W_J) = W_J w^{-1} W_I$ (i.e. $W_J w^{-1} W_I$ is the image of $W_I w W_J$ under g).

Let $y \in g(W_I w W_J)$. Then $y = g(x) = x^{-1}$ for some $x \in W_I w W_J$. Then we can write $y = ((s_{i_1} \cdots s_{i_m}) w (s_{j_1} \cdots s_{j_n}))^{-1} = (s_{j_n} \cdots s_{j_1}) w^{-1} (s_{i_m} \cdots s_{i_1})$ for some $m, n \in \mathbb{N}$, $i_1, \ldots, i_m \in I$, and $j_1 \ldots, j_n \in J$. Then $y \in W_J w^{-1} W_I$ by definition.

If $y \in W_J w^{-1} W_I$ then $y = (s_{j_n} \cdots s_{j_1}) w^{-1} (s_{i_m} \cdots s_{i_1}) = ((s_{i_1} \cdots s_{i_m}) w (s_{j_1} \cdots s_{j_n}))^{-1}$ for some $m, n \in \mathbb{N}$, $i_1, \ldots, i_m \in I$, and $j_1 \ldots, j_n \in J$. Taking $x = (s_{i_1} \cdots s_{i_m}) w (s_{j_1} \cdots s_{j_n}) \in W_I w W_J$, we see that $y = g(x)$ and thus $y \in g(W_I w W_J)$.

Our goal is now to show that G is a bijection. We first verify that G maps into $C_{w^{-1}}$. Let $D \in C_w$. Then $w \in D$ so we can write $D = W_I w W_J$ for some $I, J \subseteq \{1, 2, \ldots, n-1\}$. Since w is the minimal length element in $W_I w W_J$, we have $I \subseteq \mathrm{Asc}_L(w)$ and $J \subseteq \mathrm{Asc}_R(w)$. Then $I \subseteq \mathrm{Asc}_R(w^{-1})$ and $J \subseteq \mathrm{Asc}_L(w^{-1})$. This means w^{-1} is the minimal length element in $W_J w^{-1} W_I = G(D)$, and thus $G(D) \in C_{w^{-1}}$.

To see that G is a bijection, notice that G is invertible with $G^{-1} : C_{w^{-1}} \to C_w$ also mapping a set to its image under g. We can use the same argument to show that G^{-1} maps into C_w. Let $E \in C_{w^{-1}}$. Then $w^{-1} \in E$ so we can write $E = W_J w^{-1} W_I$ for some $I, J \subseteq \{1, 2, \ldots, n-1\}$. Since w^{-1} is the minimal length element in $W_J w^{-1} W_I$, we have $J \subseteq \mathrm{Asc}_L(w^{-1})$ and $I \subseteq \mathrm{Asc}_R(w^{-1})$. Then $J \subseteq \mathrm{Asc}_R(w)$ and $I \subseteq \mathrm{Asc}_L(w)$. This means w is the minimal length element in $W_I w W_J = G^{-1}(E)$, and thus $G^{-1}(E) \in C_w$.

It is obvious that for any $D \in C_w$, $G^{-1}(G(D)) = D$ and for any $E \in C_{w^{-1}}$, $G(G^{-1}(E)) = E$ (if we take a set of permutations and invert each element twice, we clearly end up with the same set). □

Lemma 2. *Let $f : S_n \to S_n$ be given by $f(x) = w_0 x w_0$. Then the presentation (I, w, J) is lex-minimal if and only if $(w_0(I), f(w), w_0(J))$ is. Thus f induces a bijection between C_w^* and $C_{f(w)}^*$ so that $c_w = c_{f(w)}$. This bijection is given explicitly by $F : C_w^* \to C_{f(w)}^*$ with $F((I, w, J)) = (w_0(I), f(w), w_0(J))$.*

Proof. Since f and w_0 are involutions, it is sufficient to prove that $(w_0(I), f(w), w_0(J))$ is lex-minimal whenever (I, w, J) is lex-minimal.

We will first show that f preserves lengths in that $\ell(w) = \ell(f(w))$ for all $w \in S_n$. We will do this by proving that for all permutations $w \in S_n$, (i, j) is an inversion of w if and only if $(w_0(j), w_0(i))$ is an inversion of $f(w)$. Again, since f and w_0 are involutions we only need to prove one direction.

Suppose (i, j) is an inversion of w. Then $i < j$ and $w(i) > w(j)$. Then $w_0(j) < w_0(i)$. Notice that

$$\begin{aligned} w(j) < w(i) &\iff w_0(w(j)) > w_0(w(i)) \\ &\iff w_0(w(w_0(w_0(j)))) > w_0(w(w_0(w_0(i)))) \\ &\iff f(w)(w_0(j)) > f(w)(w_0(i)). \end{aligned}$$

We now have $w_0(j) < w_0(i)$ and $f(w)(w_0(j)) > f(w)(w_0(i))$, thus $(w_0(j), w_0(i))$ is an inversion of $f(w)$. It follows that the map $(i,j) \mapsto (w_0(j), w_0(i))$ is a bijection between inversions of w and inversions of $f(w)$, and thus $\ell(w) = \ell(f(w))$ for all $w \in S_n$.

Next we show that $W_{w_0(I)} f(w) W_{w_0(J)}$ is the image of $W_I w W_J$ under f. Observe that

$$W_{w_0(I)} = \langle s_{w_0(i)} \mid i \in I \rangle = \langle w_0 s_i w_0 \mid i \in I \rangle = w_0 W_I w_0.$$

The last equality here comes from the fact that $w_0 = w_0^{-1}$. That is, we can write any element $y \in \langle w_0 s_i w_0 \mid i \in I \rangle$ as

$$y = (w_0 s_{i_1} w_0)(w_0 s_{i_2} w_0) \cdots (w_0 s_{i_m} w_0) = w_0 s_{i_1} s_{i_2} \cdots s_{i_m} w_0.$$

Then for any $x \in W_{w_0(I)} f(w) W_{w_0(J)} = (w_0 W_I w_0)(w_0 w w_0)(w_0 W_J w_0)$ we can write

$$x = (w_0 p w_0)(w_0 w w_0)(w_0 q w_0) = w_0 p w q w_0 = f(pwq)$$

for some $p \in W_I$ and $q \in W_J$, and thus $W_{w_0(I)} f(w) W_{w_0(J)} \subseteq f(W_I w W_J)$.

Similarly, if we let $x \in f(W_I w W_J)$ then there exist $p \in W_I$ and $q \in W_J$ such that $x = f(pwq) = w_0 p w q w_0 = (w_0 p w_0)(w_0 w w_0)(w_0 q w_0)$ and thus $x \in (w_0 W_I w_0)(w_0 w w_0)(w_0 W_J w_0) = W_{w_0(I)} f(w) W_{w_0(J)}$ by definition. We have shown that $W_{w_0(I)} f(w) W_{w_0(J)} \subseteq f(W_I w W_J)$ and $f(W_I w W_J) \subseteq W_{w_0(I)} f(w) W_{w_0(J)}$, therefore $W_{w_0(I)} f(w) W_{w_0(J)} = f(W_I w W_J)$.

We now return to our original goal. Let (I, w, J) be a lex-minimal presentation. Then w is the minimal length element in $W_I w W_J$ and if $W_I w W_J = W_{I'} w' W_{J'}$ then either $|I| < |I'|$ or $|I| = |I'|$ and $|J| \leq |J'|$. Since $W_{w_0(I)} f(w) W_{w_0(J)} = f(W_I w W_J)$ and f preserves lengths, $f(w)$ is the minimal length element in $W_{w_0(I)} f(w) W_{w_0(J)}$.

Now suppose $W_{w_0(I)} f(w) W_{w_0(J)} = W_{I'} w' W_{J'}$. Then $f(W_{I'} w' W_{J'}) = W_{w_0(I')} f(w') W_{w_0(J')} = W_I w W_J$. Since w_0 is a bijection, $|w_0(I)| = |I|$, $|w_0(J)| = |J|$, $|w_0(I')| = |I'|$, and $|w_0(J')| = |J'|$. Then by lex-minimality of (I, w, J), either $|w_0(I)| = |I| < |w_0(I')| = |I'|$ or $|w_0(I)| = |I| = |w_0(I')| = |I'|$ and $|w_0(J)| = |J| \leq |w_0(J')| = |J'|$. Then $(w_0(I), f(w), w_0(J))$ is lex-minimal by definition. □

Combining the two lemmas above, we obtain the following.

Corollary 1. *For all permutations $w \in S_n$,*

$$c_w = c_{w^{-1}} = c_{w_0 w w_0} = c_{w_0 w^{-1} w_0}.$$

We now consider the equivalence relation on S_n given by $w \sim z$ if $z \in \{w, w^{-1}, w_0 w w_0, w_0 w^{-1} w_0\}$. Let $[w]$ denote the equivalence class of $w \in S_n$ with respect to this relation,

$$[w] = \{w, w^{-1}, w_0 w w_0, w_0 w^{-1} w_0\}.$$

For now we will refer to these as WXML equivalence classes.

Theorem 1. *The number of WXML equivalence classes in S_n is*

$$\frac{1}{4}\left(n! + \left(2\left\lfloor \frac{n}{2}\right\rfloor\right)!! + 2 + 2\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} \prod_{j=0}^{k-1} \binom{n-2j}{2}\right).$$

Proof. Consider the identity $e : S_n \to S_n$ and the functions f and g as defined earlier. Notice that for any $w \in S_n$,

$$f(g(w)) = f(w^{-1}) = w_0 w^{-1} w_0 = (w_0 w w_0)^{-1} = g(w_0 w w_0) = g(f(w))$$

so that $G = \{e, f, g, fg\}$ is a group under function composition (it is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$). Consider also the group action $\pi : G \times S_n \to S_n$ given by $\pi(h, w) = h(w)$. By Burnside's Lemma [2, Theorem 17.1],

$$\#\{[w] \mid w \in S_n\} = \#\{\mathrm{Orb}(w) \mid w \in S_n\} = \frac{1}{4}\sum_{h \in G} |\mathrm{Fix}_h(S_n)|$$

where $\mathrm{Fix}_h(S_n) = \{w \in S_n \mid h(w) = w\}$. This means we can count the number of WXML equivalence classes by counting the number of fixed points of each function e, f, g, fg.

The identity e fixes every element in S_n so $|\mathrm{Fix}_e(S_n)| = |S_n| = n!$.

To count the number of permutations fixed by f, first note that

$$\begin{aligned} w = w_0 w w_0 &\iff w(x) = w_0(w(w_0(x)))\ \forall x \in \{1, 2, \dots, n\} \\ &\iff w(x) = n + 1 - w(n + 1 - x)\ \forall x \in \{1, 2, \dots, n\} \\ &\iff w(n + 1 - x) = n + 1 - w(x)\ \forall x \in \{1, 2, \dots, n\}. \end{aligned}$$

Let us now construct a permutation $w \in S_n$ that is fixed by f. Suppose first that n is odd. Then since $\frac{n+1}{2} = n + 1 - \frac{n+1}{2}$, it is necessary by the equivalence above that $w\left(\frac{n+1}{2}\right) = \frac{n+1}{2}$. This leaves us with $n - 1$ choices for $w(1)$. Once $w(1)$ is chosen, we are forced to set $w(n) = n + 1 - w(1)$. This leaves us with $n - 3$ choices for $w(2)$. Continuing in this manner until we reach $\frac{n+1}{2}$, we find that there are a total of $(n-1) \cdot (n-3) \cdot \dots \cdot 4 \cdot 2 = (n-1)!!$ choices. If n is even we follow the same procedure, the only difference being there is no "middle number" $\frac{n+1}{2}$ that is predetermined before we make any choices. It follows that there are $n \cdot (n-2) \cdot (n-4) \cdot \dots \cdot 4 \cdot 2 = n!!$ choices if n is even. In summary, the number of elements fixed by f is equal to $n!!$ if n is even and $(n-1)!!$ if n is odd. This can be written more compactly as $|\mathrm{Fix}_f(S_n)| = (2\lfloor \frac{n}{2} \rfloor)!!$.

Next we count the number of fixed points of g. It is a well-known result in elementary group theory that the involutions in S_n are precisely products of disjoint transpositions. To construct an involution, we first make the (mutually exclusive) choice of how many transpositions to use. Call this choice k. If n is even then $0 \leq k \leq \frac{n}{2}$, and if n is odd then $0 \leq k \leq \frac{n-1}{2}$. It is more convenient to write $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ for all $n \in \mathbb{N}$. Once k is fixed, there are $\binom{n}{2}$ choices for the first transposition. Then since the transpositions are required to be disjoint, there are $\binom{n-2}{2}$ choices for the next transposition. We proceed until we have chosen all k disjoint transpositions, which gives us a current total of

$$\prod_{j=0}^{k-1} \binom{n-2j}{2}$$

choices for each fixed k. Since disjoint cycles commute, order does not matter so we have over counted by a factor of $k!$. Putting all this information together, we obtain

$$|\mathrm{Fix}_g(S_n)| = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} \prod_{j=0}^{k-1} \binom{n-2j}{2}.$$

Lastly, observe that that $|\mathrm{Fix}_{fg}(S_n)| = |\mathrm{Fix}_g(S_n)|$, as the map $w \mapsto w_0 w$ is a bijection between $\mathrm{Fix}_g(S_n)$ and $\mathrm{Fix}_{fg}(S_n)$:

$$\begin{aligned} w \in \mathrm{Fix}_g(S_n) &\iff w = w^{-1} \\ &\iff w_0 w = w_0 w^{-1} \\ &\iff w_0 w = w_0 w^{-1} w_0 w_0 \\ &\iff w_0 w = w_0 (w_0 w)^{-1} w_0 \\ &\iff w_0 w \in \mathrm{Fix}_{fg}(S_n). \end{aligned}$$

We then sum each of these terms and divide by 4 to obtain the formula above. □

Prior to our work, this sequence ($a_n =$ the number of WXML equivalence classes in S_n) was not in the OEIS. It has since been added at A300931.

5 Future Goals

Throughout this quarter we have been writing code with the goal of implementing the main formula in [1] for counting the number of parabolic double cosets in S_n. This code is not yet finished. We would also like to write a program to visualize the w-ocean of a permutation $w \in S_n$ (a w-ocean is a diagram that represents the combinatorial structure of w and can be used to compute c_w). Another achievable goal is to characterize the sets $\{w \in S_n \mid c_w = k\}$ for small k. It would also be valuable to investigate what these sets look like for larger values of k.

References

[1] S. C. Billey, M. Konvalinka, T. K. Petersen, W. Slofstra, and B. E. Tenner. Parabolic double cosets in Coxeter groups. *Electr. J. Comb.*, 25:P1.23, 2018.

[2] J. B. Fraleigh. *A First Course in Abstract Algebra.* Addison-Wesley Pub. Co., 7 edition, 2003.