Washington Experimental Mathematics Lab

Counting K-tuples in Discrete Sets

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Integer Points

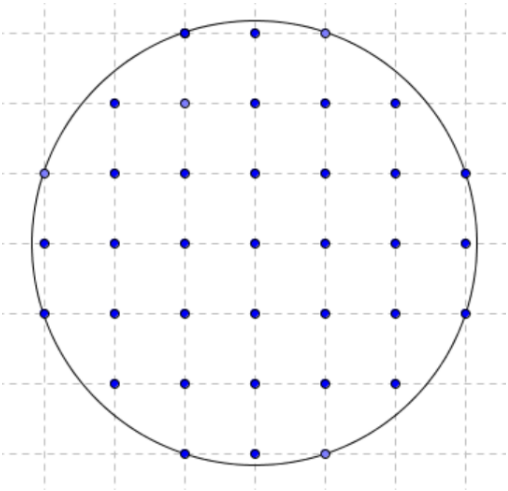


Figure: Integer lattice within the circle $x^2 + y^2 \leq 10$

Primitive Points

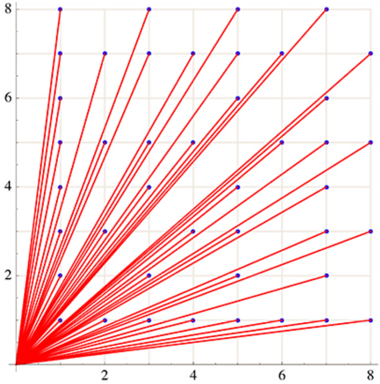
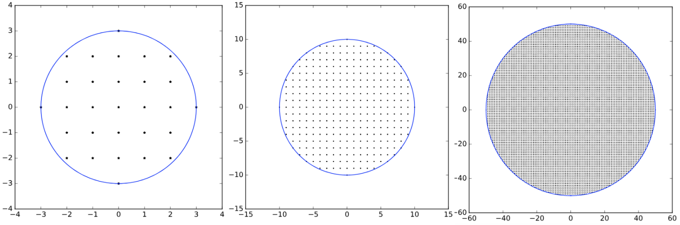


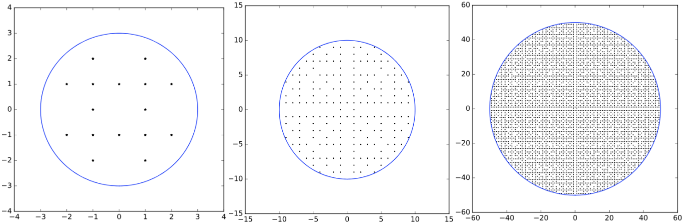
Figure: Primitive points in the first quadrant of an integer lattice

"visible points," i.e. $gcd(x, y) = 1$

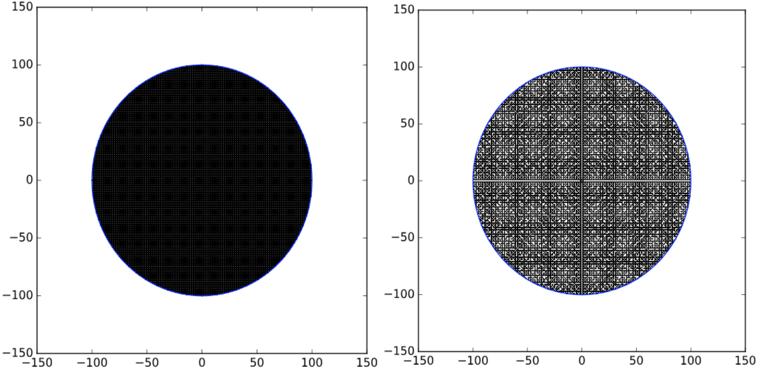
Density of Integer Points



Density of Primitive Points



Comparing Densities



Pairs

Definition

Let $Count(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in Z$$

$$gcd(a, c) = 1, \quad gcd(b, d) = 1$$

(i.e. (a, c) and (b, d) are both primitive points/vectors)

$SL_2(\mathbb{Z})$

$SL_2(\mathbb{Z})$: set of 2×2 matrices with determinant 1 and all integer entries

$$SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1 \text{ and } a, b, c, d \in \mathbb{Z} \right\}$$

$SL_2(\mathbb{Z})$ Orbits

Example:

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & a+3b \\ c & c+3d \end{pmatrix} : \mathrm{a}, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & 2a+3b \\ c & 2c+3d \end{pmatrix} : \mathrm{a}, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$SL_2(\mathbb{Z})$ Orbits Continued

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & a+3b \\ c & c+3d \end{pmatrix} : ad - bc = 1 \right\}$$
$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & 2a+3b \\ c & 2c+3d \end{pmatrix} : ad - bc = 1 \right\}$$

If $SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ Then

$$2a + 3b = a + 3b \quad \Rightarrow \quad a = 0$$

$$2c + 3d = c + 3d \quad \Rightarrow \quad c = 0$$

Since $a = c = 0$, $ad - bc = 0$ which is a contradiction to our definition that $ad - bc = 1$.

Future goals

- Decompose primitive triples into $SL_2\mathbb{Z}$ orbits
- Write a monster program that will count the density of any k-tuple of vectors given any number of determinants as input
- Write our beautiful paper with all the theory behind our findings