University of Washington

Washington Experimental Mathematics Lab

Counting K-Tuples in Discrete Sets

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March 31, 2018

1 Introduction

We were first interested in counting the density of integer lattice points in a ball of radius R. Using a python program, we found that this number converges to the area of the ball. Intuitively, one can imagine each lattice point as a pixel on a screen. The larger the radius, the more pixels you can fit inside, and thus, the closer the number of pixels will be to the actual area.

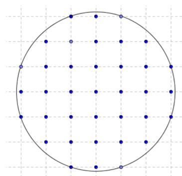

Figure 1: Circle with Lattice

We then started looking as the set of primitive points (m, n), where m and n are both coprime integers. That is, the set of points where $gcd(m, n) = 1$. Our interest was in finding a relationship between the number of pairs of vectors (m, n) in a ball of radius R with determinant k and the area the ball itself.

Geometrically, the determinant is the area of the parallelogram formed by two vectors.

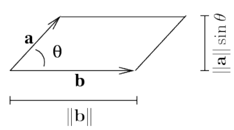

Figure 2: Geometric Depiction of Determinant

Mathematically we know the determinant is:

$$\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = m_1 n_2 - n_1 m_2 = k$$

To gather empirical evidence of a convergence for $\frac{Count(R,k)}{R^2}$ as R increases, we used python to create a $Count(R, K)$ function which counts the number of integer vector pairs within a ball of radius R that have determinant K.

As stated in the box below, $Count(R, k)$ counts the number of integer vector pairs in a ball of radius R that have a determinant k.

Let $Count(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in Z$$

$$gcd(a, c) = 1, \quad gcd(b, d) = 1$$

.

Our code implements a Farey Tree to make computing $Count(R, k)$ more efficient. It is based off of the Farey sequence. The Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which when in lowest terms have denominators less than or equal to n, arranged in order of increasing size. This helps us generate the set of coprime pairs

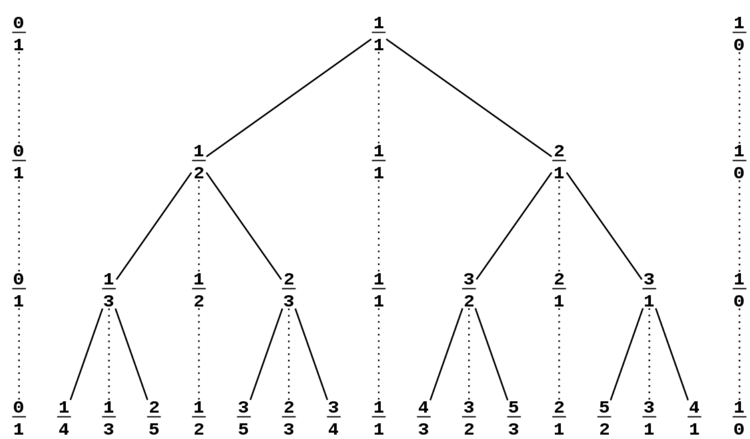

Figure 3: Farey Tree

We calibrated our code against the theorem[1] that states

$$\lim_{R \to \infty} \frac{Count(R, 1)}{R^2} = 6$$

[1]Counting Modular Matrices with Specified Euclidean Norm, Morris Newman

2 Hecke Triangle Groups

All elements in Hecke Triangle Group H_n are constructed using the following matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}$$

for n=2,3,... and where $\lambda_n = 2cos(\frac{\pi}{n})$, Multiplying every combination of S, T, S^{-1}, and T^{-1} gives all of the elements in Hecke Triangle Group H_n. By taking the first column of each of the elements in H_n, we get a set of all the vectors within the triangle group.
Previously, we were working in Hecke Triangle Group H_3. Looking at different triangle groups, we need to use a different method to build the list of vectors from which we will count pairs of fixed determinant. Construction of the Farey Tree in triangle group H_n follows this pattern:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v_j = \lambda_n v_{j-1} - v_{j-2}$$

Beginning with vectors v_1 and v_n, we again take each pair of adjacent vectors in the tree and add to the tree new linear combinations of those two vectors. The number of new vectors added to the tree at each iteration depends on the Hecke Triangle Group you are working in. For example:
In H_4, take adjacent vectors v_1 and v_2 and add the vectors $\sqrt{2}v_1 + v_2$ and $v_1 + \sqrt{2}v_2$ to the tree.

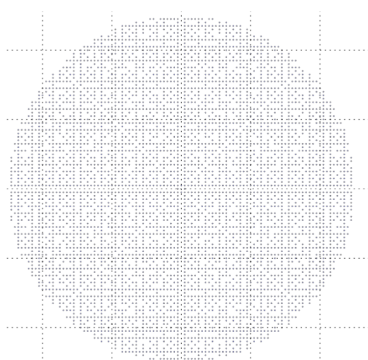

Figure 4: H_3 in a circle

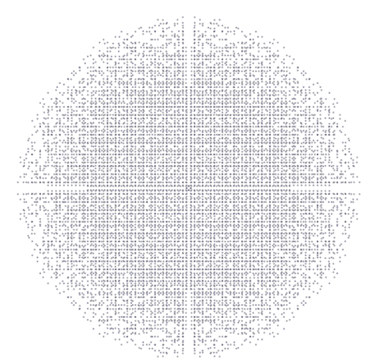

Figure 5: H_4 in a circle

3.1 Previous Research

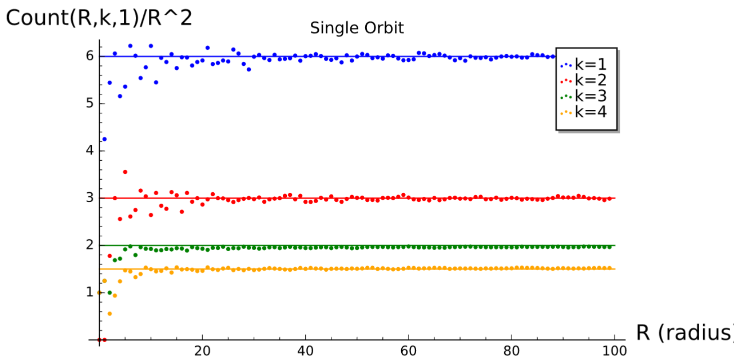

Figure 6: Single orbit $\dfrac{Count(R,k,1)}{R^2} \rightarrow \dfrac{6}{k}, \quad \forall k \in \mathbb{Z}$

The first graph shows the result of our code for increasing R and values $k = 1, 2, 3, 4$. Each $Count(R, k)$ can be split up into a certain number of groups or "orbits" based on the determinant k. This graph just looks at a single orbit for each $Count(R, k)$, which we see converges to $\dfrac{6}{k}$ as R increases.

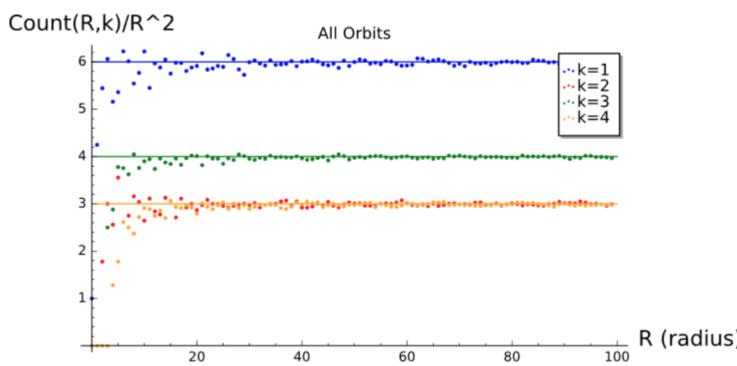

Figure 7: All orbits $\dfrac{Count(R,k)}{R^2} \rightarrow \dfrac{6\varphi(k)}{k}, \quad \forall k \in \mathbb{Z}$

The second graph shows the total $Count(R, k)$ for $k = 1, 2, 3, 4$. We see that each of the orbits contributes $6k$ to the total, and that the number of orbits for a given k is equal to $\varphi(k)$. Hence, when you increase R the total count of integer vector pairs with determinant k, proportional to the size of the ball (R^2), converges to $\dfrac{6\varphi(k)}{k}$.

3.2 Current Research

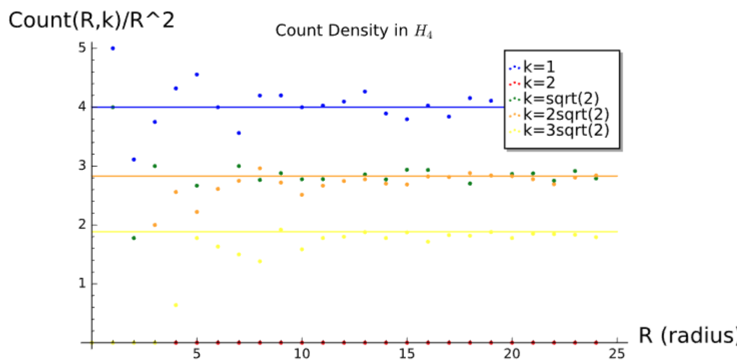

Figure 8: $\dfrac{Count(R,k)}{R^2} \to \dfrac{4\varphi_4(k)}{k}$

The graph shows the result of our code for increasing R and values

$$k = 1, 2, \sqrt{2}, 2\sqrt{2}, 3\sqrt{2}$$

Similar to the "All Orbits" graph for H_3 above, these counts, when divided by (R^2), converge to a constant over k, times a phi function. This phi function $\varphi_4(k)$ is a pseudo-euclidean algorithm, which outputs an integer value based on the input k.

4 Triples

Having looked at pairs, our next goal was to investigate triples in different groups. A triple now consists of 3 vectors. Instead of determining whether a single pair of vectors has a determinant k and fits inside a ball of size R, we are comparing 3 different pairs of vectors by coupling each unique set of vectors in the triple. Let us define the determinant of the first pair of vectors to be k_1, the second pair to be k_2, and the third pair to be k_3. For instance, consider the following vectors:

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\det(A, B) = k_1, \quad \det(A, C) = k_2, \quad \det(B, C) = k_3$$

In order to efficiently compute the count of triples we have determined a relationship between the determinants k_1, k_2, k_3. If we know k_1 and k_2 we can use the following function to compute k_3:

$$k_3 = \frac{k_2(a_1 b_2 + a_2 b_1) - k_1(a_1 c_2 + a_2 c_1)}{2a_1 a_2} \tag{1}$$

An issue for determining k_3 arises when either a_1 or a_2 is zero. Since the farey tree construction does not contain the zero vector, we consider a special case where only one of the vector components equals zero. Without loss of generality, consider the case $a_2 = 0$ and $a_1 \neq 0$. We can derive the following alternate equation that considers this case:

$$k_3 = \frac{b_1 k_2}{a_1} - \frac{k_1 c_1}{a_1} \tag{2}$$

5 Future Goals

In our continuation of this project next quarter, we hope to explore these ideas further by finding the counts for triples across various Hecke Triangle Groups by adjusting our farey tree code. Our goal is use our computations to generalize our findings to k-tuples.