Rotation Random Walk

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1 Problem Statement

We are interested in the following stochastic process:

Algorithm 1 Rotation Random Walk

Fix $\alpha \in (0, 1)$ to be the step size.
Set $Y_0 = 0$, where $Y_i \in [0, 1)$ is the position of the random walk at time i.
for $i = 1, 2, ...n$ **do**

$$X_i = \begin{cases} +1, & \text{w.p } \frac{1}{2} \\ -1, & \text{w.p } \frac{1}{2} \end{cases}$$

$Y_i = \{Y_{i-1} + \alpha X_i\}$ the new position mod 1.

$$Z_i = \begin{cases} +1, & \text{if } Y_i \in [0, \frac{1}{2}) \\ -1, & \text{if } Y_i \in [\frac{1}{2}, 1) \end{cases}$$

Question: The partial sums $S_n = \sum_{i=1}^n Z_i$ define a new stochastic process on $\mathbb{Z}$. Does this sum $\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i$ follow a central limit theorem (the Z_i's are not independent!)?

2 Rational Case proof

We have that for $\alpha = \frac{p}{k}$ where $\gcd(p, k) = 1$, the possible positions of Y_i segment S^1 into k equidistant portions since $\{n\alpha\} = 0 \Leftrightarrow k|n$. Thus without loss of generality we can consider $\alpha = \frac{1}{k}$.
The idea of our argument is to split up our $\frac{1}{n} \sum_{i=1}^n Z_i$ into independent portions and then apply the central limit theorem to the sum of iid random variables.

Definition 2.1. *Fix $k \in \mathbb{N}$. Suppose we have a random walk on k "pegs" equal length apart along the unit circle. The return time T_k is the random variable which is the number of steps until we return to the starting point, where we take a left and right step with equal probability.*

Lemma 2.1. *The return time has finite expected value:*

$$\mathbb{E}[T_k] = k$$

Proof:
We do a first step-analysis using the law of total expectation, conditioning on whether we take a left or right step. Let T_k be the random variable that denotes the first return time to the starting point when there are k positions.

$$\mathbb{E}[T_k] = \mathbb{E}[T_k|R]\mathbb{P}(R) + \mathbb{E}[T_k|L]\mathbb{P}(L) = \frac{1}{2}[\mathbb{E}[T_k|L] + \mathbb{E}[T_k|R]] = \mathbb{E}[T_k|R]$$

$$\mathbb{E}[T_k|R] = \frac{1}{2}(\mathbb{E}[T|R^2] + \mathbb{E}[T|RL]) = \frac{1}{2}\mathbb{E}[T_k|R^2] + 1$$

$$\mathbb{E}[T_k|R^2] = \frac{1}{2}(\mathbb{E}[T_k|R^3] + \mathbb{E}[T_k|R^2L]) = \frac{1}{2}(\mathbb{E}[T_k|R^3] + 2 + \mathbb{E}[T_k|R])$$

Plugging this back in, we get:

$$\mathbb{E}[T_k] = \frac{1}{2}\mathbb{E}[T_k|R^2] + 1 = \frac{1}{3}\mathbb{E}[T_k|R^3] + 2$$

From this we get the general pattern:

$$\mathbb{E}[T_k] = \frac{1}{j}\mathbb{E}[T_k|R^j] + (j-1)$$

which is equivalent to

$$j(\mathbb{E}[T_k] - (j-1)) = \mathbb{E}[T_k|R^j]$$

This pattern is verified by the following inductive argument
We have already established the base cases of $i = 1, 2$. For the inductive argument we again use the law of total expectation

$$\mathbb{E}[T_k|R^i] = \frac{1}{2}\left(\mathbb{E}[T_k|R^{i+1}] + (2 + \mathbb{E}[T_k|R^{i-1}])\right)$$

using the inductive hypothesis

$$i(\mathbb{E}[T_k] - (i-1)) = \frac{1}{2}\left(\mathbb{E}[T_k|R^{i+1}] + (2 + (i-1)(\mathbb{E}[T_k] - (i-1))\right)$$

$$2i\mathbb{E}[T_k] - 2i^2 + 2i = \mathbb{E}[T_k|R^{i+1}] + 2 + i\mathbb{E}[T_k] - \mathbb{E}[T_k] - i^2 + 2i - 1$$

rearranging the equation yields the desired result

$$(i+1)(\mathbb{E}[T_k] - i) = \mathbb{E}[T_k|R^{i+1}]$$

It is the case that $\mathbb{E}[T_k|R^k] = k$ since taking k right steps from the origin gets you back to the origin, and thus

$$\mathbb{E}[T_k] = \frac{1}{k}k + (k-1) = k$$

Lemma 2.2. *Let N be a nonnegative integer-valued random variable with mean γ, and $X_1, ..., X_N$ be a random number of iid rvs each with mean μ. Further suppose X_i and N are independent for all i. Let $X = \sum_{i=1}^N X_i$. We have that*

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}[N]\ \mathbb{E}[X_i] = \gamma\mu$$

Proof:

First, notice that

$$\mathbb{E}[X \mid N = n] = \mathbb{E}\left[\sum_{i=1}^N X_i \mid N = n\right] = \mathbb{E}\left[\sum_{i=1}^n X_i\right] = n\mu$$

By the law of total expectation,

$$\mathbb{E}[X] = \sum_n \mathbb{E}[X \mid N = n]\mathbb{P}(N = n) = \sum_n n\mu \cdot \mathbb{P}(N = n) = \mu \sum_n n \cdot \mathbb{P}(N = n) = \mu \cdot \mathbb{E}[N] = \gamma\mu$$

Lemma 2.3. *For $k \in \mathbb{N}$, consider the rotation random walk by $\alpha = 1/k$ given in algorithm 1, and let $a \in \{0, ..., k-1\}$. Then, for k^2 sufficiently large, $\mathbb{P}(Y_i \neq 0\ \forall i \leq k^2 \mid Y_0 = a) \leq 0.9$.*

Proof:

Reformulate the problem as a random walk starting at a, with boundaries 0 and k. Then, the probability we desire is the same as the probability we never leave this range of $[0, k]$ taking steps of size 1 after k^2 steps. Let $(X_n)_{n\in\mathbb{N}}$ be iid Rademacher random variables, and $S_n = a + \sum_{i=1}^n X_i$ be the random walk. For n large enough, $S_n = a + \sum_{i=1}^n X_i \xrightarrow{d} \mathcal{N}(a, n)$.

$$\mathbb{P}(0 < S_i < k \ \forall 1 \le i \le k^2) \le \mathbb{P}(0 < S_{k^2} < k) \le \mathbb{P}(a - k < S_{k^2} < a + k) \approx \Phi(1) - \Phi(-1) \approx 0.683 < 0.9$$

The main Theorem:

Theorem 2.4. *The random walk defined by partial sums of Z_i follows a central limit theorem.*

$$\frac{\frac{1}{n}\sum_{i=1}^n Z_i - \mu}{\sigma/\sqrt{n}} \to \mathcal{N}(0,1)$$

Proof: Fix $\alpha \in \mathbb{Q} \cap (0,1)$, WLOG we say $\alpha = \frac{1}{k}$ for some $k \in \mathbb{N}$. Let $R_i = \inf\{j : Y_j = 0, j > R_{i-1}\}$ be the time of the i^{th} return to the origin (let $R_0 = 0$). Notice that $\mathbb{E}[R_i - R_{i-1}] = \mathbb{E}[T_k] = k$, and that $R_0, R_1 - R_0, R_2 - R_1, \ldots$ are iid. Fix $n \in \mathbb{N}$, and let $r_n = \max\{m : R_m \le n\}$ be the number of times we visit the origin after n steps. Then,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{r_n}\left(\sum_{j=R_{i-1}}^{R_i - 1} Z_j\right) + \frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{r_n} W_i + \frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i$$

where

$$W_i = \sum_{j=R_{i-1}}^{R_i - 1} Z_j$$

.

Let $t_n = \mathbb{E}[r_n]$, so we can further break down the sum as

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{t_n} W_i + \frac{1}{\sqrt{n}}\sum_{i=t_n+1}^{r_n} W_i + \frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i$$

Then $W_1, \ldots, W_{r_n}$ are iid since each increment $R_i - R_{i-1}$ are return times to the origin, in which the position random variable (Y_i) resets back to zero.

We show that (1) the first term converges in distribution to a normal, (2) the second term converges in distribution/probability to 0, (3) the third term converges in distribution/probability to 0, and (4) By Slutsky's Theorem we can conclude that our sum converges to a normal distribution

(1): We will show that $t_n \to \infty$ as $n \to \infty$, and then we can just apply the CLT.

For large enough j, Stirling's formula says $j! \sim \sqrt{2\pi j}\frac{j^j}{e^j}$, and hence $\binom{2j}{j} = \frac{(2j)!}{j!j!} \sim \frac{\sqrt{4\pi j}(2j)^{2j}/e^{2j}}{2\pi j \cdot j^{2j}/e^{2j}} = \frac{4^j}{\sqrt{j\pi}}$.

First, we lower bound the probability we are at the origin after $2j$ steps. Notice that if we have exactly j steps in each direction, we are guaranteed to be back at the origin. Hence,

$$\mathbb{P}(\text{at origin after } 2j \text{ steps}) \ge \binom{2j}{j}\left(\frac{1}{2}\right)^{2j} \sim \frac{1}{\sqrt{j\pi}}$$

For $i = 1, \ldots, n$, let V_i be 1 if we are at the origin at time i, and 0 otherwise. Then, we can write $r_n = \sum_{i=1}^n V_i$. Hence

$$t_n = \mathbb{E}[r_n] = \sum_{j=1}^n \mathbb{E}[V_j] \ge \sum_{j=1}^{n/2}\mathbb{E}[V_{2j}] = \sum_{j=1}^{n/2}\mathbb{P}(V_{2j} = 1) \ge \sum_{i=1}^{n/2}\binom{2j}{j}\left(\frac{1}{2}\right)^{2j} \sim \sum_{i=1}^{n/2}\frac{1}{\sqrt{j\pi}} \ge \sum_{i=1}^{n/2}\frac{1}{\sqrt{n\pi}} = \frac{n}{2}\frac{1}{\sqrt{n\pi}} = \frac{\sqrt{n}}{2\sqrt{\pi}}$$

So as $n \to \infty$, $t_n \to \infty$, and the CLT applies to this first term.

(2): We are going to show $\frac{1}{\sqrt{n}} \sum_{i=t_n+1}^{r_n} W_i \xrightarrow{p} 0$. This quantity is large in absolute value if and only if at least one of the following happens:

1. $|r_n - t_n|$ is large
2. $|W_i|/\sqrt{n}$ is large

We show that these two probabilities can be bounded for n sufficiently large, and then the union bound will imply that the quantity is $< \epsilon$.

For the first, we show that the following (equivalent to $r_n - t_n$ being close) holds with high probability:

$$R_{t_n - c\sqrt{n}} \leq n \leq R_{t_n + c\sqrt{n}}$$

We bound the probability of one side not happening, and the other side is equivalent. $t_n = \mathbb{E}[r_n] \approx n/k$ and $\mathbb{E}[T_i] = k$, so $\mathbb{E}[\sum_{i=1}^{t_n - c\sqrt{n}} T_i] = (n/k - c\sqrt{n})(k) = n - kc\sqrt{n}$. Let $Var(T_i) = \alpha < \infty$, and so $\sigma^2 = Var(\sum_{i=1}^{n-c\sqrt{n}} T_i) = \alpha(n/k - c\sqrt{n})$. Hence

$$\mathbb{P}(R_{t_n - c\sqrt{n}} > n) = \mathbb{P}\left(\sum_{i=1}^{t_n - c\sqrt{n}} T_i > n\right) = \mathbb{P}\left(\sum_{i=1}^{t_n - c\sqrt{n}} T_i - (n - kc\sqrt{n}) > kc\sqrt{n}\right)$$

Applying Chebyshev's inequality gives

$$\leq \frac{\alpha(n/k - c\sqrt{n})}{k^2 c^2 n} < \frac{\alpha}{k^3 c^2} < \epsilon$$

.

Hence we can always find c such that this probability is $< \epsilon$.

For the second, we need to bound the probability of $|W_i|/\sqrt{n}$ being large. We bound the probability of $W_i/\sqrt{n}$ being large, and the same bound holds for $-W_i/\sqrt{n}$ being largely negative by symmetry.

$$\mathbb{P}(W_i/\sqrt{n} > \epsilon) = \mathbb{P}(W_i > \epsilon\sqrt{n}) = \mathbb{P}\left(\sum_{i=1}^{R_1} Z_i > \epsilon\sqrt{n}\right) = \sum_{j=1}^{n} \mathbb{P}\left(\sum_{i=1}^{R_1} Z_i > \epsilon\sqrt{n} \mid R_1 = j\right) \mathbb{P}(R_1 = j)$$

If $j < \epsilon/\sqrt{n}$, the probability is 0, so

$$= \sum_{j=\epsilon/\sqrt{n}}^{n} \mathbb{P}\left(\sum_{i=1}^{R_1} Z_i > \epsilon\sqrt{n} \mid R_1 = j\right) \mathbb{P}(R_1 = j) = \sum_{j=\epsilon/\sqrt{n}}^{n} \mathbb{P}\left(\sum_{i=1}^{j} Z_i > \epsilon\sqrt{n}\right) \mathbb{P}(R_1 = j)$$

By Markov's inequality, $\mathbb{P}(R_1 \geq u) \leq \frac{\mathbb{E}[R_1]}{u} = \frac{k}{u}$ Hence the probability $R_1 = k$ for large k decays linearly since the probability you don't return after such a long time is 0 by the inequality. So for $j > \epsilon/\sqrt{n}$ as in our sum, $\mathbb{P}(R_1 = j) \leq \mathbb{P}(R_1 \geq j) \leq \frac{k}{j}$. So we have

$$\leq k \sum_{j=\epsilon/\sqrt{n}}^{n} \mathbb{P}\left(\sum_{i=1}^{j} Z_i > \epsilon\sqrt{n}\right) / j$$

. But this probability term is going to 0 by Chebyshev's inequality; this is the probability Z_i is getting far from its mean of 0. Hence this goes to 0.

(3): When considering $\frac{1}{\sqrt{n}} \sum_{i=R_{r_n}}^{n} Z_i$, we wish to show

$$\frac{1}{\sqrt{n}} \sum_{i=R_{r_n}}^{n} Z_i \xrightarrow{d} 0$$

This is equivalent to showing for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i\right| > \epsilon\right) = 0$$

since $|Z_i| = 1$ we have that

$$\left|\frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i\right| \leq \frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} |Z_i| = \frac{n-R_{r_n}}{\sqrt{n}}$$

In other words

$$\left|\frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i\right| > \epsilon \Rightarrow \frac{n-R_{r_n}}{\sqrt{n}} > \epsilon$$

so

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i\right| > \epsilon\right) \leq \mathbb{P}\left(\frac{n-R_{r_n}}{\sqrt{n}} > \epsilon\right) = \mathbb{P}\left(R_{r_n} < n - \epsilon\sqrt{n}\right)$$

$$= \sum_{j=1}^{n-\epsilon\sqrt{n}} \mathbb{P}\left(R_{r_n} = j\right)$$

Since R_{r_n} was defined as the last return time of the Y_is,

$$\mathbb{P}(R_{r_n} = j) = \mathbb{P}(\{Y_j = 0\} \cap \{Y_i \neq 0\, \forall i > j\}) = \mathbb{P}(Y_i \neq 0\, \forall i > j \mid Y_j = 0)\mathbb{P}(Y_j = 0) \leq \mathbb{P}(Y_i \neq 0\, \forall i > j \mid Y_j = 0)$$

This probability is equivalent to the probability of a new random walk on the circle not returning after n steps. From Lemma 2.3, for sufficiently large n we get

$$\mathbb{P}(R_{r_n} = j) \leq .9^{(n-j)/k^2 - 1}$$

Thus

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=R_{r_n}}^{n} Z_i\right| > \epsilon\right) \leq \sum_{j=1}^{n-\epsilon\sqrt{n}} .9^{(n-j)/k^2 - 1}$$

This the tail of a geometric series with some reindexing

$$= \sum_{j=\epsilon\sqrt{n}}^{n} .9^{(j)/k^2 - 1} \leq \sum_{j=\epsilon\sqrt{n}}^{\infty} .9^{(j)/k^2 - 1} \to 0 \text{ as } n \to \infty$$

(4): **(Slutsky)** Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$, with (X_n) and (Y_n) independent sequences of random variables. Then, $X_n + Y_n \xrightarrow{d} X + c$.
We are given that

$$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x) = F_X(x)$$

and

$$\lim_{n\to\infty} \mathbb{P}(Y_n \leq y) = \mathbb{P}(c \leq y) = \mathbb{1}_{(c\leq y)}$$

We wish to show that

$$\lim_{n\to\infty} \mathbb{P}(X_n + Y_n \leq z) = \mathbb{P}(X + c \leq z) = \mathbb{P}(X \leq z - c)$$

We suppose X_n is discrete, but a similar argument holds if X_n is continuous by replacing sums with integrals and $\mathbb{P}(X_n = x)$ by $f_{X_n}(x)dx$.

$$
\begin{aligned}
\mathbb{P}(X_n + Y_n \le z) &= \sum_x \mathbb{P}(X_n + Y_n \le z | X_n = x)\mathbb{P}(X_n = x) \\
&= \sum_x \mathbb{P}(Y_n \le z - x)\mathbb{P}(X_n = x) \\
&\to \sum_x \mathbb{1}_{(c \le z - x)}\mathbb{P}(X_n = x) \\
&= \sum_{x : x \le z - c} \mathbb{P}(X_n = x) \\
&= \mathbb{P}(X_n \le z - c) \\
&\to \mathbb{P}(X \le z - c)
\end{aligned}
$$

as desired. A similar argument holds replacing sums with integrals.
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