Washington Experimental Mathematics Lab

Stability Spectrum for PDEs

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Stability Spectrum for PDEs

Motivation To determine the stability of periodic solutions to certain PDEs, including the focusing mKdV equation

Problem To determine the eigenvalues of a linear operator such that the associated eigenfunctions are bounded

Methods Taking advantage of the periodicity of the coefficients using Floquet theory and Fourier series

Example

Let $\mathcal{L} = -\partial_x^2$ and consider the eigenvalue problem

$$\mathcal{L}y = \lambda y, \qquad \lambda \in \mathbb{C},$$

or equivalently, $y'' + \lambda y = 0$.

We say $\lambda \in \sigma(\mathcal{L})$ if the associated $y(x)$ is bounded for all $x \in \mathbb{R}$, $\lim_{|x|\to\infty} |y(x)| < \infty$.

Example (ct'd.)

To find $\sigma(\mathcal{L})$, we solve the differential equation $y'' + \lambda y = 0$. The general solution is

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}x}.$$

In general, $\sqrt{-\lambda} \in \mathbb{C}$, so let $\sqrt{-\lambda} = \alpha + \mathrm{i}\beta$ for some $\alpha, \beta \in \mathbb{R}$.

Example (ct'd.)

Letting $\sqrt{-\lambda} = \alpha + \mathrm{i}\beta$, rewrite $y = c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}x}$ as

$$\begin{aligned} y &= c_1 \mathrm{e}^{(\alpha+\mathrm{i}\beta)x} + c_2 \mathrm{e}^{-(\alpha+\mathrm{i}\beta)x} \\ &= c_1 \mathrm{e}^{\alpha x}\mathrm{e}^{\mathrm{i}\beta x} + c_2 \mathrm{e}^{-\alpha x}\mathrm{e}^{-\mathrm{i}\beta x} \\ &= c_1 \mathrm{e}^{\alpha x}[\cos(\beta x) + \mathrm{i}\sin(\beta x)] + c_2 \mathrm{e}^{-\alpha x}[\cos(\beta x) - \mathrm{i}\sin(\beta x)] \end{aligned}$$

We require that the solution $\lim_{|x|\to\infty} |y(x)| < \infty$. Therefore, $\alpha = 0$.

Example (ct'd.)

With $\alpha = 0$,

$$\begin{aligned} y &= c_1[\cos(\beta x) + \mathrm{i}\sin(\beta x)] + c_2[\cos(\beta x) - \mathrm{i}\sin(\beta x)] \\ &= A\cos(\beta x) + \mathrm{i}B\sin(\beta x) \end{aligned}$$

is bounded for any $\beta \in \mathbb{R}$. $\sqrt{-\lambda} = \mathrm{i}\beta$ only if $\lambda \geq 0$.
So $\sigma(-\partial_{xx}) = [0, \infty)$.

Numerically computed spectrum

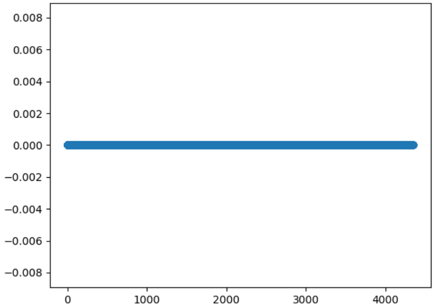

Figure: Spectrum of the operator $\mathcal{L} = -\partial_x^2$.

Pictures

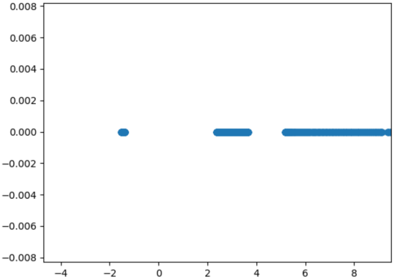

Figure: Spectrum of the operator $\mathcal{L} = -\partial_x^2 + 2q\cos(2x)$ for $q = 2$ from 0 to 10.

Future goals

Next steps Determining the spectrum of

$$\mathcal{L} = -\partial_y^3 + (c - 6U^2)\partial_y + 12UU_y,$$

which is used in determining the stability of the solution $u(x,t) = U(y) = U(x - ct)$ of the focusing mKdV equation:

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

Challenges Understanding elliptic functions and generalizing our code.