

Parabolic Double Cosets in Symmetric Groups

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Permutations

- A *permutation* is a rearrangement of things.
- {stop, spot, tops, pots, post} are permutations of the letters o,p,s,t.
- The following are **all** permutations of {1, 2, 3}:

[123]	[213]
[132]	[312]
[231]	[321]

- We call this set of $3! = 6$ permutations the *symmetric group* on 3 elements, and we denote it by S_3 .

Parabolic cosets

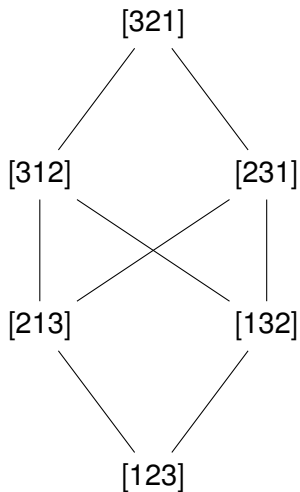
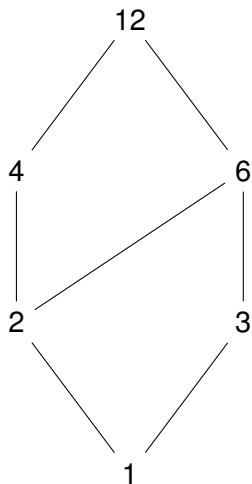
- Which permutations can you make from $[43172864]$ by only swapping numbers of the same color?
- Which permutations can you make from $[\underline{43}17\underline{28}64]$ by only swapping numbers underlined by the same color?
- The set of all such permutations are examples of left/right *parabolic cosets*.
- What if we allow both types of swapping?

Parabolic double cosets

- If we allow swapping of certain adjacent **values and** certain adjacent positions, we end up with a *parabolic double coset*.
- Example: $\{[2\underline{3}514], [2\underline{4}513], [2\underline{5}413], [2\underline{5}314]\}$ is a parabolic double coset in S_5 .

Partial orders

Examples:



Questions

- How many parabolic double cosets are in S_n ?
- What are the most efficient ways of counting them?
- Which Bruhat intervals are parabolic double cosets (what makes them special, if anything)?

Counting them by their:

- “smallest” element
- structure in Bruhat order
- rank (length of “largest” – length of “smallest” element)
- cardinality

- Let k be a fixed natural number. The number of parabolic double cosets in S_n of cardinality k is equal to a factorial times a polynomial (as a function of n).
- We have computed such functions for $k = 1, 2, 4, 6$.

Future goals

- Compute the number of parabolic double cosets in S_n for $n = 14, 15, \dots$
- Further classify which types of intervals are and aren't parabolic double cosets.
- Explore other methods of enumeration (based on minimal and maximal elements).