Uncertainty Quantification

Numerical Analysis Research Club
Autumn Quarter, 2011

Department of Applied Mathematics
University of Washington

http://depts.washington.edu/uwnarc/

UQ motivation

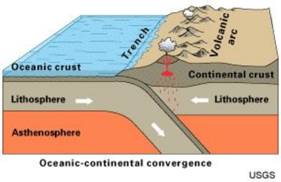
Numerical simulation of a physical (or other) process
typically gives one solution.

But there are often many sources of uncertainty:

- Unknown physical parameters or data,
  - Epistemic uncertainty: There is a true value but we can't measure it exactly. May only have upper and lower bounds.
  - Aleatoric uncertainty: Not a single set of data of interest. Instead a known or estimated probability distribution.
- The mathematical equations are only a model of reality,
- The numerical solution does not exactly solve the model equations.

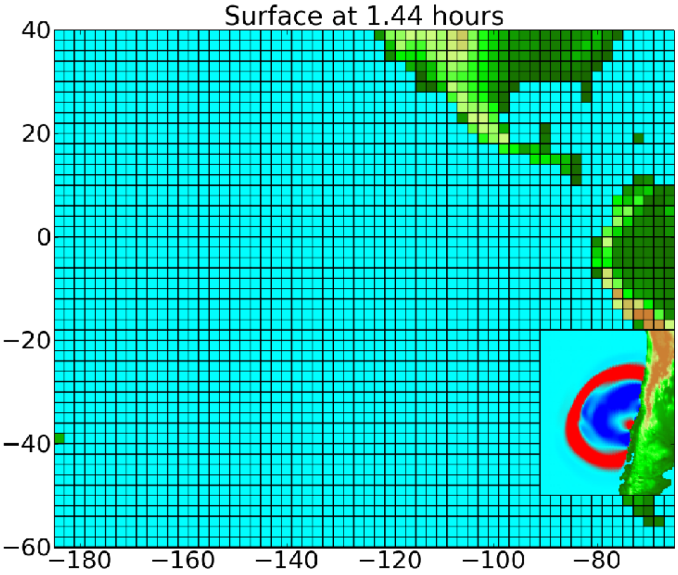
Related themes: Sensitivity analysis, *A posteriori* error estimation, Estimation of reliability.

Tsunamis caused by subduction zone earthquakes



- Small amplitude in ocean (< 1 meter) but can grow to 10s of meters at shore.
- Run-up along shore can inundate 100s of meters inland
- Long wavelength (as much as 200 km)
- Propagation speed $\sqrt{gh}$ (much slower near shore)
- Average depth of Pacific or Indian Ocean is 4000 m $\Longrightarrow$ average speed 200 m/s $\approx$ 450 mph

27 February 2010 tsunami



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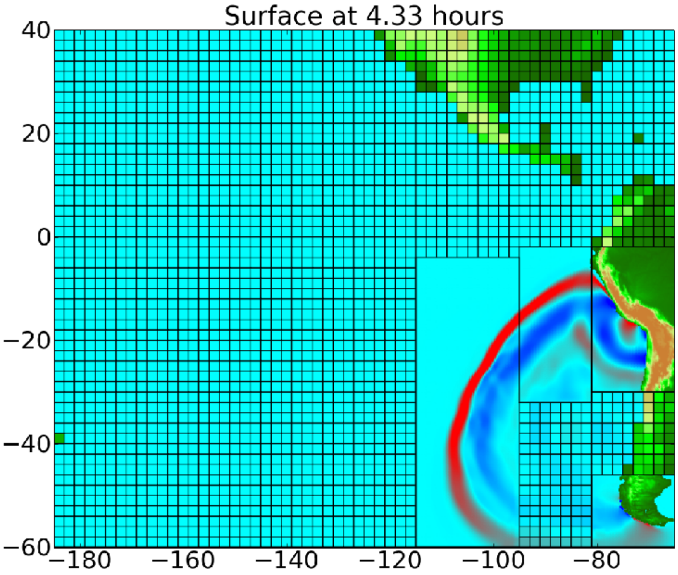

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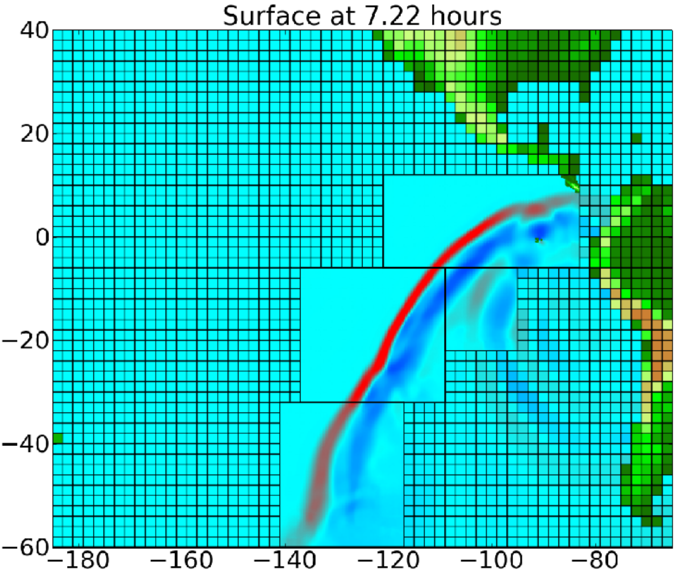

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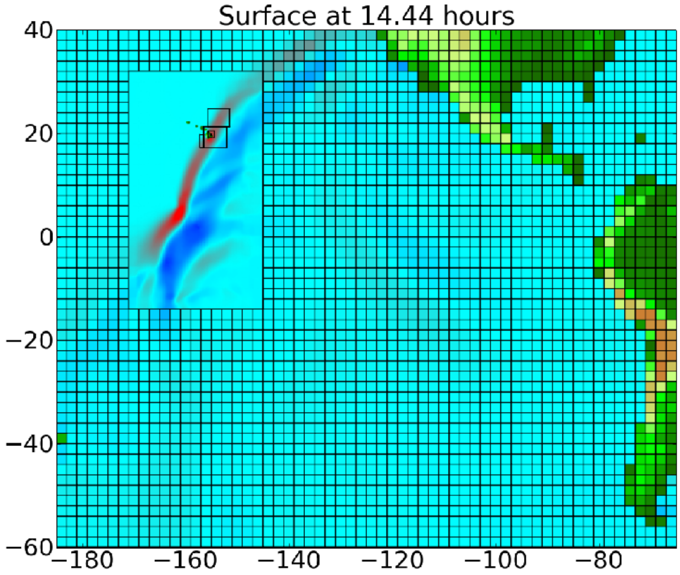

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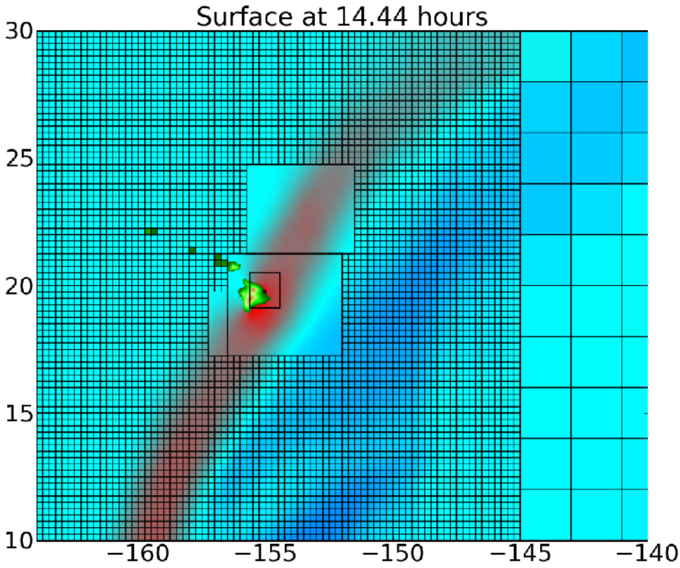

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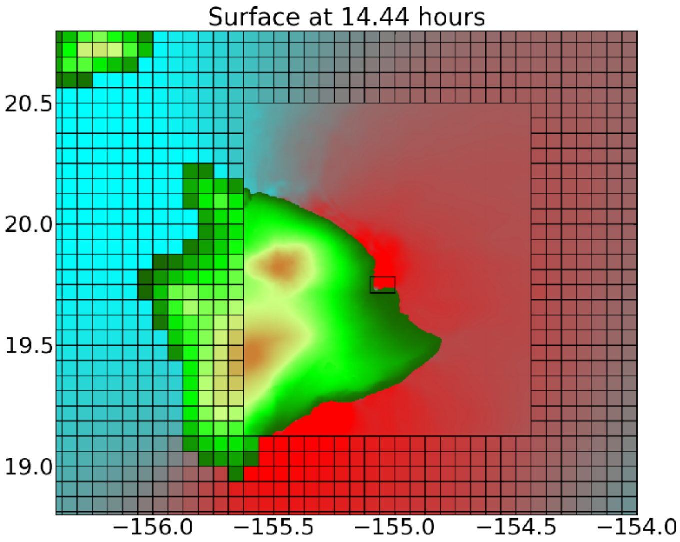


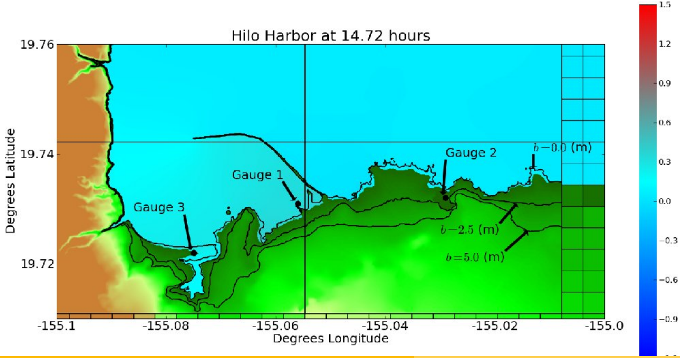
Inundation of Hilo, Hawaii

Using 5 levels of refinement with ratios 8, 4, 16, 32.

Resolution ≈ 160 km on Level 1 and ≈ 10m on Level 5.

Total refinement factor: $2^{14} = 16,384$ in each direction.

With 15 m displacement at fault:



Inundation of Hilo, Hawaii

Using 5 levels of refinement with ratios 8, 4, 16, 32.

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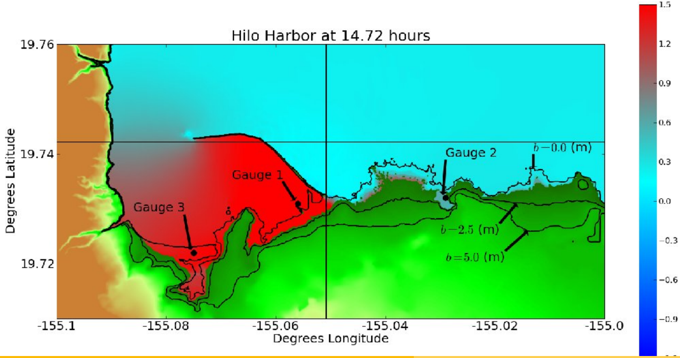
Total refinement factor: $2^{14} = 16,384$ in each direction.

With 90 m displacement at fault:



PTHA Based on PSHA: Methodology & Applications

Frank González
U. Washington -- Earth & Space Sciences

NRC/USGS Workshop on Landslide Tsunami Probability

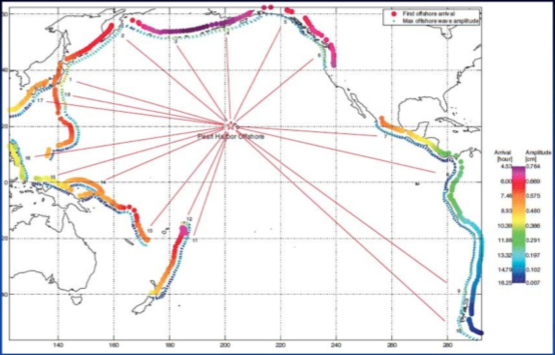
August 18-19, 2011
Woods Hole, Massachusetts

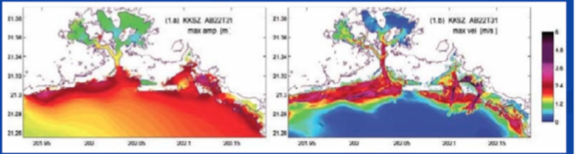
F. I. González, UW/ESS

Sensitivity Analysis

Site-specific response to many scenarios

Tang, et al., 2006.





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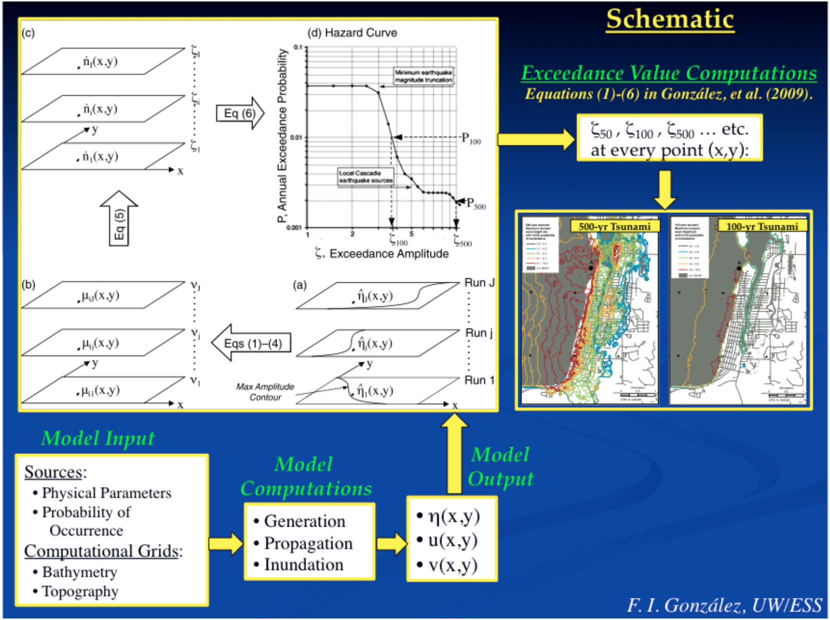
Schematic
(c)
(d) Hazard Curve
Eq (6)
Eq (5)
Minimum earthquake magnitude truncation
Local Cascadia earthquake sources
P, Annual Exceedance Probability
ζ, Exceedance Amplitude
(b)
(a)
Run J
Run j
Run 1
Eqs (1)–(4)
Max Amplitude Contour
Exceedance Value Computations
Equations (1)-(6) in González, et al. (2009).
ζ50 , ζ100 , ζ500 … etc.
at every point (x,y):
500-yr Tsunami
100-yr Tsunami
Model Input
Sources:
• Physical Parameters
• Probability of Occurrence
Computational Grids:
• Bathymetry
• Topography
Model Computations
• Generation
• Propagation
• Inundation
Model Output
• η(x,y)
• u(x,y)
• v(x,y)
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Is the planned coverage adequate ?
Are there priority areas not covered ?

F. I. González, UW/ESS

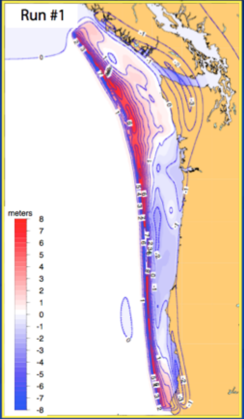
Must Update this Table for CC PTHA Study

Table 1. Source Specification for Earthquakes Used in This Study[a]

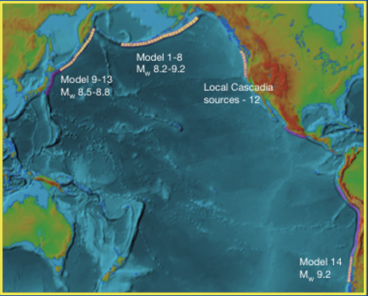
Source	Location	M	Length (km)	Width (km)	Slip (m)	T_M (y)
1	AASZ	9.2	1000	100	17.7	1,313
2	AASZ	9.2	1000[c]	100	17.7[c]	750
3	AASZ	9.2	600	100	Dist.	750
4	AASZ	9.2	1200	100	14.8[c]	1,133
5	AASZ	9.2	1200	100	14.8	750
6	AASZ	8.2	300	100	2.1	875
7	AASZ	8.2	300	100	2.1	661
8	AASZ	8.2	300	100	2.1	661
9	KmSZ	8.8	500	100	9.8	100
10	KmSZ	8.8	500	100	9.8	100
11	KrSZ	8.5	300	100	5.8	500
12	KrSZ	8.5	300	100	5.8	500
13	KrSZ	8.5	300	100	5.8	500
14	SChSZ	9.5	1100[c]	100	40.0	300
15	CSZ	9.1	1100[c]	Variable	Variable	520

(T_M is mean interevent time)

Near-Field detail (1 of 12)



All Sources: Far- & Near-field



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UQ techniques

Some possibilities, adapted from Tim Barth's slides,

www.stanford.edu/group/cits/workshop/tutorials.html

Statistical methods:

- Monte Carlo simulation,
- Stratified sampling,
- Latin hypercube sampling,
- Response surface method,
- Multi-level Monte Carlo (MLMC).

Recast as deterministic equation for probability distribution:

- Perturbation expansion of random fields,
- Stochastic operator expansions,
- (Generalized) Polynomial Chaos,
- Adjoint equations.

ODE model problem

Solve $u'(t) = -u(t)$ for $u(T)$ given $u(0) = \eta$.

Exact solution: $u(t) = e^{-T}\eta$.

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Suppose η is a random variable, uniformly distributed in $[\eta_1,\ \eta_2]$.

What is probability distribution of $u(T)$?

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Exact solution: $u(t) = e^{-T}\eta$.

Suppose η is a random variable, uniformly distributed in $[\eta_1,\ \eta_2]$.

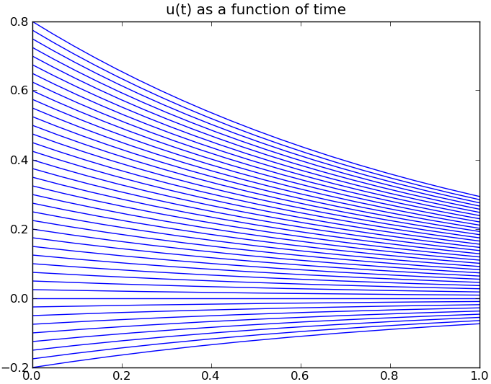
What is probability distribution of $u(T)$?

Answer: uniformly distributed in interval $[e^{-T}\eta_1,\ e^{-T}\eta_2]$.

ODE model problem

$u'(t) = -u(t)$ for $u(T)$ given $u(0) = \eta \implies u(t) = e^{-T}\eta$.

η is a random variable, uniformly distributed in $[-0.2,\ 0.8]$.



Equally spaced values of η give equally spaced trajectories!

ODE model problem

$u'(t) = -u(t)$ for $u(T)$ given $u(0) = \eta \implies u(t) = e^{-T}\eta$.

Probability density function $f_T(v)$:

$$P[u_1 < u(T) < u_2] = \int_{u_1}^{u_2} f_T(v)\,dv$$

Initial uniform distribution in $[\eta_1,\ \eta_2]$:

$$f_0(v) = \begin{cases} 1/(\eta_2 - \eta_1) & \text{if } \eta_1 < v < \eta_2, \\ 0 & \text{otherwise} \end{cases}.$$

ODE model problem

To compute $f_T(v)$:

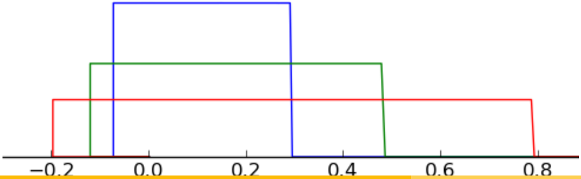
$$P[u_1 < u(T) < u_2] = P[e^T u_1 < \eta < e^T u_2]$$

$$\int_{u_1}^{u_2} f_T(v)\,dv = \int_{e^T u_1}^{e^T u_2} f_0(v)\,dv$$

If $f_0(v) = 1/(\eta_2 - \eta_1)$ is constant in the interval then RHS reduces to $e^T(u_2 - u_1)/(\eta_2 - \eta_1)$, so

$$f_T(v) = \begin{cases} e^T/(\eta_2 - \eta_1) & \text{if } e^T\eta_1 < v < e^T\eta_2, \\ 0 & \text{otherwise} \end{cases} .$$

At $t = 0,\ 0.5,\ 1$:

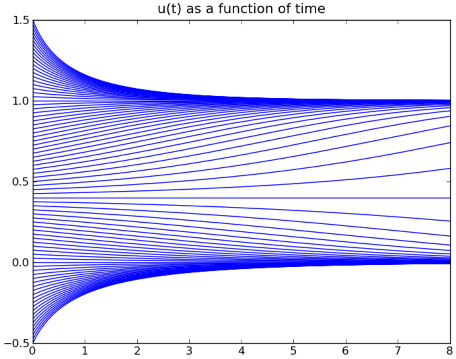


ODE model problem II

Solve $u'(t) = u(t)(u(t) - 0.4)(1 - u(t))$ for $u(T)$ given $u(0) = \eta$.

Unstable fixed point at $u = 0.4$.

All solutions converge towards one of the
Stable fixed points at $u = 0,\ 1$,

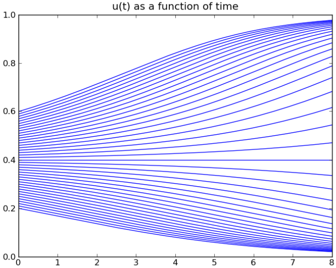


ODE model problem II

Solve $u'(t) = u(t)(u(t) - 0.4)(1 - u(t))$ for $u(T)$ given $u(0) = \eta$.

Again assume η uniformly distributed in $[\eta_1,\ \eta_2]$.

$\eta_1 = 0.2, \ \ \eta_2 = 0.6$:



Density at $t = 0,\ 4,\ 8$: