Part 1 – Understanding

1. (18 pts) Consider two circular orbits, one of radius \( \textit{r}_1 \) called \( \textit{O}_1 \) and the other of radius \( \textit{r}_2 \) called \( \textit{O}_2 \) (we will assume that \( \textit{r}_2 > \textit{r}_1 \)). This question will explore the efficiency of circular-to-circular orbital maneuvers as a function of \( \textit{r}_2/\textit{r}_1 \).

(a) A Hohmann transfer is a two-burn maneuver that can take us from orbit \( \textit{O}_1 \) to orbit \( \textit{O}_2 \) along an elliptic transfer ellipse. Show that the \( \Delta v \) required for this maneuver can be written as

\[ \frac{\Delta v}{\textit{v}_1} |_{\text{Hohmann}} = \sqrt{\frac{\textit{r}_1}{\textit{r}_2}} - 1 + \sqrt{\frac{\textit{r}_2}{\textit{r}_1}} \frac{2}{\textit{r}_1 + \frac{2}{\textit{r}_2}} - \sqrt{\frac{\textit{r}_1}{\textit{r}_2}} \frac{2}{\textit{r}_1 + \frac{2}{\textit{r}_2}} \]

(b) A bi-elliptic transfer is a three-burn maneuver that achieves the same objective. In this case, we travel from orbit \( \textit{O}_1 \) to an auxiliary orbit \( \textit{O}_3 \) along an elliptic transfer orbit. Orbit \( \textit{O}_3 \) is a circular orbit with radius \( \textit{r}_3 \). Next, we travel from orbit \( \textit{O}_3 \) to orbit \( \textit{O}_2 \) along another elliptic orbit, and perform a third burn to circularize our orbit at a radius \( \textit{r}_2 \). See Figure 1 for a pictoral description – in the figure, the thick black curves are the circular orbits and the thick red lines are the portions of the two elliptic transfer orbits that would be flown. Show that the total \( \Delta v \) for this maneuver can be written as

\[ \frac{\Delta v}{\textit{v}_1} |_{\text{Bi-elliptic}} = \sqrt{\frac{\textit{r}_3}{\textit{r}_1}} \frac{2}{\textit{r}_1 + \frac{2}{\textit{r}_3}} - 1 + \sqrt{\frac{\textit{r}_1}{\textit{r}_3}} \frac{1}{\textit{r}_3} \frac{\textit{r}_1}{\textit{r}_3} \frac{2}{\textit{r}_1 + \frac{2}{\textit{r}_3}} \frac{\textit{r}_3}{\textit{r}_1} + \sqrt{\frac{\textit{r}_1}{\textit{r}_3}} \frac{2}{\textit{r}_3} \frac{\textit{r}_1}{\textit{r}_3} \frac{2}{\textit{r}_1 + \frac{2}{\textit{r}_3}} \frac{\textit{r}_3}{\textit{r}_1} - \frac{\textit{r}_1}{\textit{r}_2} \]

(c) For \( \textit{r}_3/\textit{r}_1 = 100 \), plot \( \frac{\Delta v}{\textit{v}_1} \) versus \( \textit{r}_2/\textit{r}_1 \) for both maneuvers on the same set of axes. Show at least \( \textit{r}_2/\textit{r}_1 \in [1, 60] \).

(d) For \( \textit{r}_3/\textit{r}_1 \in \{5, 50, 100, 200, 500, 1000, 100000\} \), compute and tabulate the points of intersection between the two \( \frac{\Delta v}{\textit{v}_1} \) curves. What do you notice as \( \textit{r}_3/\textit{r}_1 \to \infty \)?

(e) Using the results from (c) and (d), interpret the data in terms of which transfer type you would use for different values of \( \textit{r}_2/\textit{r}_1 \).

2. (12 pts) Consider two masses; one of mass \( \textit{m} \) moving at velocity \( \textit{v} \) and the other of mass \( \Delta \textit{m} \) moving at velocity \( \textit{v}_o \). At time \( \textit{t} \), there is a force \( \textit{F} \) acting on mass \( \textit{m} \), and the two masses are separate. At time \( \textit{t} + \Delta \textit{t} \), the two masses have merged into one entity of mass \( \textit{m} + \Delta \textit{m} \) and travel at velocity \( \textit{v} + \Delta \textit{v} \). The combined system is still subject to the same force \( \textit{F} \). See Figure 2 for a depiction.

(a) Using Newton’s second law, derive an expression that relates the force \( \textit{F} \) to the time rate of change of velocity \( \textit{v} \). We will define \textit{thrust} as \( \textit{T} = - (\textit{v} - \textit{v}_o) \frac{\textit{d}\textit{m}}{\textit{d}\textit{t}} \).
(b) Define the relative speed as \( v_e = \|v - v_o\|_2 \). Using the expression for thrust \( T \) and your dynamics in part (a), derive an expression for the mass at some time \( t_2 \) as a function of mass at some time \( t_1 \) and the \( \Delta V \) imparted only by thrust over this time span.

(c) If \( r_2/r_1 = 10 \) and \( r_3/r_1 = 500 \), how much propellant mass is consumed during the Hohmann and bi-elliptic transfers if the satellite initially weighs 1000 kg? How about 10000 kg?

3. (16 pts) The atmospheric drag acting on a low-earth orbit satellite can be approximated by

\[
F_a = -\frac{1}{2} c_D A \rho v^2 = -Dv,
\]

where \( c_D \) is the drag coefficient, \( A \) is a cross-sectional area and \( \rho \) is the atmospheric density. The force per unit mass can be expressed as \( f_a = F_a/m \).

(a) Show that the effect of atmospheric drag of the satellite’s semi-major axis is

\[
\frac{da}{dt} = -2D \frac{a^2 v^2}{\mu}.
\]

(b) Show that the effect of atmospheric drag on the eccentricity is

\[
\frac{de}{dt} = D \frac{1 - e^2}{e} \left( 1 - \frac{v^2}{\mu} \right).
\]
(c) If the atmospheric drag does not affect the eccentricity of a circular orbit, what does the result in part (a) tell you about the spacecraft’s orbit?

(d) A common model of atmospheric density is to assume that it decreases exponentially with radial distance. You may notice that your Molniya[1] orbit from homework 1 had both a low-altitude portion and a high-altitude portion. For these highly elliptic orbits, we can approximate the effect of atmospheric drag as a tangential $\Delta v$ at perigee of each orbit. What will be the long-term effect of atmospheric drag on the Molniya orbit you designed?

(e) How many orbits will it take before the altitude at apogee has changed by 1 km?

Part 2 – Design

4. (25 pts) You’re tasked with designing the trajectory and thrust sequence for placing a communications satellite in geostationary orbit. The objective is to select the launch site and maneuver sequence that minimizes the fuel required by the upper stage only (i.e., not the primary launch vehicle itself). Possible launch sites are:

<table>
<thead>
<tr>
<th>Launch Site</th>
<th>Possible Inclinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cape Canaveral, USA</td>
<td>$i \in [28, 57]^\circ$</td>
</tr>
<tr>
<td>Kourou, French Guiana</td>
<td>$i \in [5, 100]^\circ$</td>
</tr>
<tr>
<td>Baikonur, Kazahkstan</td>
<td>$i \in [49, 99]^\circ$</td>
</tr>
</tbody>
</table>

(a) A geosynchronous orbit is characterized as a circular orbit whose period matches the period of Earth’s rotation[2]. At what distance from the center of the Earth must a geosynchronous satellite orbit?

(b) A geostationary orbit is a geosynchronous orbit with zero inclination. If you were to look outside (from UW) and spot a geostationary satellite, what would you see it doing? What about a general geosynchronous satellite?

(c) Choose a launch site and initial orbit inclination from above. Your primary launch vehicle places you in an elliptic insertion orbit with a perigee radius of 180 km and an apogee radius equal to the answer from part (a). The first maneuver you must do is to “zero the inclination”. Compute the $\Delta v$ required to zero the inclination from your chosen initial orbit.

(d) The second maneuver will circularize your orbit at the geostationary radius. What is the required $\Delta v$, when should you thrust, and in what direction?

(e) Because you are clever, you decide you should combine the maneuvers from parts (c) and (d) into a single burn that achieves both objectives. Can this be done? If so, what is the required $\Delta v$? Compare this answer to the sum of the $\Delta v$ computed in parts (c) and (d).

(f) What is the minimum amount of fuel (in kg) you must carry to achieve a transfer to geostationary orbit if the satellite itself weighs 1200 kg and you need an extra 250 kg of fuel for future station-keeping throughout the mission?

(g) Simulate the combined maneuver and provide some meaningful visuals to showcase your orbit control design. Show off the initial and terminal orbits for at least one complete orbit each and be sure to indicate where the thrust maneuver takes place.

Part 3 – Feedback

This part is optional: Provide some constructive feedback on the course so far. Some specific examples may be the lectures (e.g., too fast, too slow) or the homeworks (e.g., too easy, too hard). Helpful comments can make the course better going forward!

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[1] The Molniya orbits (Russian for “lightning”) were designed in the 60s and used for communications with high latitude regions, of which Russia has plenty. Their need comes from the high relative cost – or even infeasibility – of using geostationary satellites for communication in these regions.

[2] The rotation rate of Earth is provided in our constant sheet on the course webpage.