**Homework 3**

AA 528 – Spacecraft Dynamics and Control

Due: January 31, 2019 at 10:00am

**Part 1 – Understanding**

1. (8 pts) We saw in class that a general rotation from a frame $F_1$ to another frame $F_2$ can be described by a (unit) vector $a$ and an angle $\phi$. We also saw that this rotation can be described by a direction cosine matrix, $C_{21}$, and that the coordinates of the vector $a$ are invariant under the action of this rotation so that $a_1 = a_2$. This question will obtain an expression for $C_{21}$ as a function of $a_1$ (or $a_2$) and $\phi$.

   (a) A general vector $v$ has a component that is parallel to $a$ and a component that is orthogonal to $a$ so that $v = v_{\|} + v_{\perp}$. Write expressions for $v_{\|}$ and $v_{\perp}$ using only $v$ and $a$.

   (b) If we rotate $v$ about $a$, only the component $v_{\perp}$ will change. Let us call this rotated vector $v_{\text{new}}$, and its (new) component that is orthogonal to $a$ shall be $v_{\perp,\text{new}}$. With the help of Figure 1, write $v_{\text{new}}$ as a function of $v$, $a$ and $\phi$.

   (c) We know that $v_{\text{new}} = F_{1}^{T} C_{21} v_{1}$. Use the fact that $(a \cdot v)a = F_{1}^{T}(a_{1}a_{1}^{T})v_{1}$ to obtain an expression for $C_{21}$ as a function of $a_{1}$ and $\phi$.

![Figure 1: Geometric picture that goes with problem 1(b).](image)

2. (6 pts) A principal rotation is defined as a rotation about one of the coordinate axes of a coordinate frame $F$. If $x, y, z$ are the unit vectors that define the coordinate axes of $F$, then the direction cosine matrices corresponding to a principal rotation about each axis are

   $$
   C_{x}(\theta) = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & \cos \theta & \sin \theta \\
   0 & -\sin \theta & \cos \theta
   \end{bmatrix}, \quad
   C_{y}(\theta) = \begin{bmatrix}
   \cos \theta & 0 & -\sin \theta \\
   0 & 1 & 0 \\
   \sin \theta & 0 & \cos \theta
   \end{bmatrix}, \quad
   C_{z}(\theta) = \begin{bmatrix}
   \cos \theta & \sin \theta & 0 \\
   -\sin \theta & \cos \theta & 0 \\
   0 & 0 & 1
   \end{bmatrix}.
   $$

   It is common in aerospace to use “roll-pitch-yaw” angles to describe orientation. Typically, the direction cosine matrix is computed from a yaw about the $z$-axis by an angle $\phi$, a pitch about the (transformed) $y$-axis by an angle $\theta$, and finally a roll about the (transformed) $x$-axis by an angle $\psi$. 

(a) Compute the direction cosine matrix for this rotation sequence as a function of \((\psi, \theta, \phi)\).

(b) If \(\theta = \pm 90^\circ\), what is the direction cosine matrix? Do you see anything wrong?

3. (10 pts) In Sections 2.6.2 and 2.6.3 in the book, it is shown that the axes describing the orbital plane are related to the inertial axis via a z-y-z principal rotation sequence. In homework 1 we obtained the inertial position and velocity from a set of orbital elements – we will do the opposite here.

(a) Write a function (in the programming language of your choice) that computes the eccentric anomaly from the mean anomaly. Show that for an orbit with eccentricity \(e = 0.1\) and mean anomaly \(M = \pi/4\) rad, we have \(\psi = 0.86126\) rad.

(b) Write a function (in the programming language of your choice) that computes the inertial position and velocity, \(\{r_I, v_I\}\), from a set of orbital elements \(\alpha = [a e i \Omega \omega M]^T\). Test your function with

\[
\alpha = \begin{bmatrix}
a \\
e \\
i \\
\Omega \\
\omega \\
M
\end{bmatrix} = \begin{bmatrix}
6730.954 \text{ km} \\
0.0006676 \\
54.615^\circ \\
247.463^\circ \\
130.288^\circ \\
325.277^\circ
\end{bmatrix} = \begin{bmatrix}
6730.954 \text{ km} \\
0.0006676 \\
0.953205 \\
4.319038 \\
2.273949 \\
5.677150
\end{bmatrix}.
\]

Part 2 – Design

4. (22 pts) The ISS needs a Canadarm3. In this problem you will design the rendezvous trajectory to get this valuable piece of equipment from its insertion orbit to the International Space Station (ISS). Your launch provider cannot (for safety reasons) place you exactly in the ISS orbit. As such, your spacecraft finds itself in a \(400 \times 480 \text{ km}\) insertion orbit with

\[
i = 51.77^\circ \quad \Omega = 7.6^\circ \quad \omega = 292.1673^\circ.
\]

The ISS is in a 400 km circular orbit with

\[
i = 51.64^\circ \quad \Omega = 7.7208^\circ \quad \omega = 292.1673^\circ.
\]

In this problem, we will first circularize our orbit, and then use the Hill’s equations to design a two-burn impulsive maneuver that will achieve rendezvous.

(a) We must first circularize our orbit to match that of the ISS. Compute the \(\Delta v\) vector that will place the spacecraft in a 400 km circular orbit. After circularizing, Houston tells you that you lag the ISS by \(0.3^\circ\) in mean anomaly.

(b) Use the ISS orbital parameters to compute the direction cosine matrix, \(C_{H-I}\), that maps inertial coordinates to Hill coordinates\(^\dagger\) when \(M = 0\) for the satellite. **Hint:** The coordinate directions we defined form the rows of this matrix.

(c) Using the Hill’s equations we derived in class, show that

\[
\begin{bmatrix}
\rho(t) \\
\dot{\rho}(t)
\end{bmatrix} = \begin{bmatrix}
A_{11}(t,n) & A_{12}(t,n) \\
A_{21}(t,n) & A_{22}(t,n)
\end{bmatrix} \begin{bmatrix}
\rho(t_0) \\
\dot{\rho}(t_0)
\end{bmatrix}
\]

\text{(1)}

where \(n\) is the mean-motion of the ISS orbit. Note: you may assume the matrix \(A_{12}(t,n)\) is invertible for all almost all times \(t > 0\).

(d) Let \(t_0^c\) be the instant where the spacecraft satisfies \(M = 0\) but *after* the impulsive maneuver to circularize our orbit. Compute the initial conditions \(\rho(t_0^c)\) and \(\dot{\rho}(t_0^c)\).

\(^\dagger\)The Hill frame is centered at the ISS
(e) We shall apply an impulsive thrust at time \( t_0 \) so that \( \dot{\rho}(t_0^-) \neq \dot{\rho}(t_0^+) \). For rendezvous, we want to achieve \( \rho(t_f) = 0 \) for a given final time \( t_f \). If the spacecraft evolves passively for \( t \in (t_0, t_f) \), use part (c) to compute \( \dot{\rho}(t_0^+) \) as a function of \( \rho(t_0) \). What is the \( \Delta v \) to apply at time \( t_0 \)?

(f) The second impulsive burn will occur at time \( t_f \) to match a desired final relative velocity. Let \( t_f^- \) be the instant just prior to the application of thrust. What is the relative velocity at time \( t_f^- \), \( \dot{\rho}(t_f^-) \)? You are told to rendezvous with the ISS at a relative velocity of 1 m/s in the along-track direction. What is the \( \Delta v \) to apply at time \( t_f \)?

(g) Provide the total \( \Delta v \) required of the total maneuver and plot the relative motion for \( t \in (t_0, t_f) \) — i.e., the motion seen from the ISS. Present the data in a meaningful way and be sure to: label your axes, provide a title or caption and provide a legend for any figures with more than one object!

(h) BONUS: (3 pts) Prove that the matrix \( A_{12}(t,n) \) is invertible for all \( t \) in the set

\[
\left\{ t \in \mathbb{R}_+ \mid t \neq \frac{k\pi}{n}, \forall k \in \mathbb{N} \text{ and } 8 \sin(nt) \neq 3nt(1 + \cos(nt)) \right\}.
\]

**Part 3 – Feedback**

This part is optional: Provide some constructive feedback on the course so far. Some specific examples may be the lectures (e.g., too fast, too slow) or the homeworks (e.g., too easy, too hard). Helpful comments can make the course better going forward!

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\(^2\text{Passive here means it does not apply any thrust on this open interval.}\)