Part 1 – Understanding

1. (14 pts) Consider the following three-parameter vector for representing orientation:

\[ p = \tan \frac{\phi}{2} a. \]  

(a) Show that the angular displacement \( p_3 \) equivalent to an angular displacement – or rotation – of first \( p_1 \) and then \( p_2 \) is

\[ p_3 = \frac{p_1 + p_2 + p_1^T p_2}{1 - p_1^T p_2}. \]

(b) Show that the direction cosine matrix associated with (1) is

\[ C = \frac{1 - p^T p}{1 + p^T p} I_3 \]

(c) Show that the parameter \( p \) can be found from a direction cosine matrix \( C \in SO(3) \) according to

\[ p = \frac{1}{1 + \text{tr}(C)} \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \]

where \( \text{tr}(C) \) denotes the trace operator.

(d) Do these parameters have a singularity? If so, where is it?

2. (12 pts) Earth orbiting satellites (and their operators) require knowledge of several key coordinate frames. Among them are the Earth-Centered-Inertial (ECI) frame, \( F_I \), the Earth-Centered-Earth-Fixed (ECEF) frame, \( F_F \), and the local Topocentric frame, \( F_T \). A pictorial description is given in Figure 1. We have already worked with the ECI frame in class; it is an inertial frame whose \( x \) axis points to the vernal equinox and whose \( z \) axis is aligned with Earth’s spin axis.

The simplest model of Earth’s rotation defines the ECEF frame to share its \( z \) axis with the ECI frame, but rotate with a (constant) angular rate equal to the Earth’s spin rate. The \( x \) axis always points through the prime meridian in the equatorial plane (i.e., longitude zero). We shall define \( \omega = \omega (t-t_0) \).

The Topocentric frame is centered at a location on the surface of the earth with latitude \( \delta \) and longitude \( \lambda \). We will define the \( x \) axis to be locally south, the \( y \) axis to be locally east, and the \( z \) axis to be locally up, or zenith.

(a) Find the rotation (or direction cosine) matrix that maps ECI coordinates to ECEF coordinates if the angle between \( x_I \) and \( x_F \) is assumed to be equal to Greenwich Mean Time; \( \theta_{\text{GMT}} = \omega (t-t_0) \).

(b) An observer measures the position of an object in the Topocentric frame and records the coordinates \( \rho_T = [x y z]^T \). What are the coordinates of this same measurement in the ECI frame?
(c) The observer also measures a velocity $\dot{\rho}_T$. Show that the inertial coordinates of this measurement are given by
\[ \mathbf{v}_I = C_{I-T}(\dot{\rho}_T + \omega_{\oplus}^x(\rho_T + r_S, T)) \]

3. (6 pts) Given the relationship
\[ C_{2-1} = \mathbf{q}_v^T \mathbf{q}_v^T - 2q_4 \mathbf{q}_v^T + q_4^2 I_3 + q_v q_v^T \]
and the fact that we know a generic vector’s coordinates satisfy $x_2 = C_{2-1} x_1$, write $x_2$ as a function of the unit quaternion $\mathbf{q} = [q_v^T, q_4]^T \in \mathbb{R}^4$ and $x_1$, using the multiplication rules defined in class.

**Part 2 – Design**

4. (20 pts) In this question you will design a maneuver to point a camera at a specific location on the ground. Let us assume that we’ve got a new satellite in low Earth orbit, and want to point an on-board camera at UW for a class picture. As the spacecraft orbits, it will be nominally fixed in inertial space, only rotating when commanded to do so by you.

The spacecraft has limited actuation, and for risk mitigation we impose a $5^\circ$/s slew rate limit on any maneuvers performed. In addition to the ECI and ECEF frames introduced in Problem 2 (see Figure 1), we need three new frames. First, we define the Topocentric frame to have the exact same orientation as the Topographic frame, but an origin at the center of the Earth. Next, the body frame, $\mathcal{F}_B$ is centered at the spacecraft’s center of mass, rotates rigidly with it, and has directions along the spacecraft’s principal axes. A camera frame, $\mathcal{F}_C$, is centered at the camera, and mounted so that the lens is along the $z_C = z_B$ axis, but rotated $45^\circ$ about the $z_B$ axis.

For this problem you will need the latitude and longitude of UW:
\[ \delta_{UW} = 47.659878^\circ \quad \text{and} \quad \lambda_{UW} = -122.305968^\circ. \]
(a) It is common to use the unit quaternion as the on-board state of the spacecraft. This unit quaternion represents the map from inertial to body-frame coordinates, or \( q := q_{b\rightarrow i} \). Write a function (in the programming language of your choice) that computes a quaternion from a direction cosine matrix.

(b) Ideally when you take a picture, the spacecraft should be directly overhead of us. When this is the case, the relationship between the topographic and camera frames can be easily computed. First, we know that \( z_T = -z_C \) and we will arbitrarily choose \( y_T = y_C \). What is the rotation matrix from Topocentric to Camera frame coordinates, \( C_{c\rightarrow t} \) at the time of the picture?

(c) Your maneuver will start at 10:00am GMT, and you guess that it will take 60 seconds. What is the rotation matrix mapping inertial to body frame coordinates at the end of the maneuver, \( C_{b\rightarrow i}(t_f) \)? What is the corresponding unit quaternion, \( q(t_f) \)?

(d) You know (from your navigation system) that at 10:00am GMT your attitude is \( q(t_0) = [0 0 0 1]^T \). We’ll define the error quaternion as \( q_{err} = q(t_0)^* \otimes q(t_f) \). What is the axis and angle pair that corresponds to the error quaternion?

(e) Based on your answer to part (d), can your satellite perform the required reorientation in 60 seconds subject to the slew rate limits if it rotates at a constant angular rate?

(f) Simulate the maneuver using the quaternion kinematics and your answers until this point. Present the data in a meaningful way.

BONUS: What is the least amount of time your satellite needs to perform the requisite maneuver to snap the picture, if it must move at a constant angular rate throughout the maneuver? Provide the time needed, final attitude quaternion, rotation rate, and a plot of the simulation.

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1 Please check any comments of previous homeworks to make sure you’re including axis labels, units, captions, etc!