Kronecker Product of Networked Systems and their Approximates

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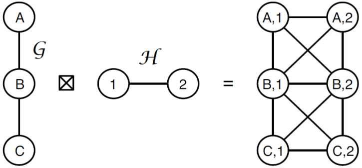
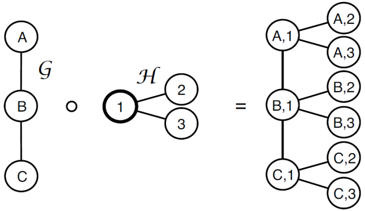
Graph Products: Networks within Networks

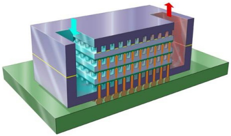
- Many ways to compose graphs $\mathcal{G}$ and $\mathcal{H}$
  - Cartesian product $\mathcal{G}\square\mathcal{H}$
  - Tensor/Direct/Kronecker product $\mathcal{G}\times\mathcal{H}$
  - Strong product $\mathcal{G}\boxtimes\mathcal{H}$
  - Lexicographic product $\mathcal{G}\bullet\mathcal{H}$
  - Rooted product $\mathcal{G}\circ\mathcal{H}$
  - Corona product $\mathcal{G}\odot\mathcal{H}$
  - Star product $\mathcal{G}\star\mathcal{H}$
- How does modularity of the network manifest itself as modularity within the state dynamics?



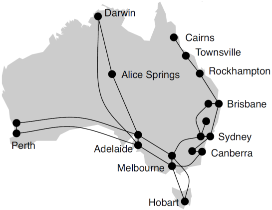


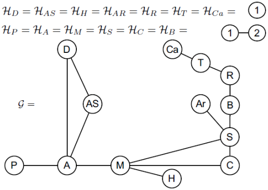
Kronecker Product: $(\textbf{Graphs}, \times) \to (\textbf{Dynamics}, \otimes)$

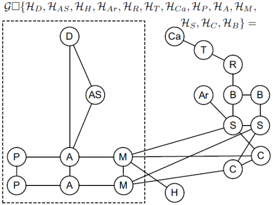
Graph Product Examples



- Periodic Structures:
  e.g., hypercube multiprocessors, building trusses
- Compartmental Networks:
  e.g., air traffic networks, chemical reactions
- Constant degree expander graphs:
  e.g., computer networks, sorting networks, cryptography







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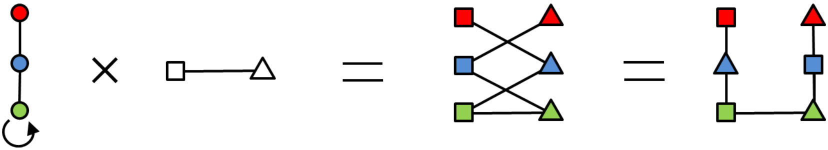
Cite: Parsonage *et al.* “Generalized Graph Products for Network Design and Analysis”, 2011.

Graph Kronecker Product

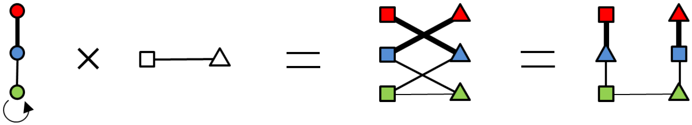
- Kronecker product $\mathcal{G} \times \mathcal{H}$
- Vertex set: $V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$
- Edge set: $(x_1, x_2) \sim (y_1, y_2)$ is in $\mathcal{G} \times \mathcal{H}$
  - if $x_1 \sim y_1$ and $x_2 \sim y_2$



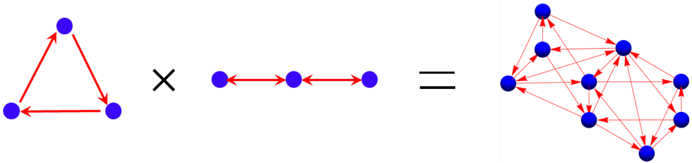
Algebraic Representation

$$\mathcal{A}(\mathcal{G} \times \mathcal{H}) = \mathcal{A}(\mathcal{G}) \otimes \mathcal{A}(\mathcal{H})$$

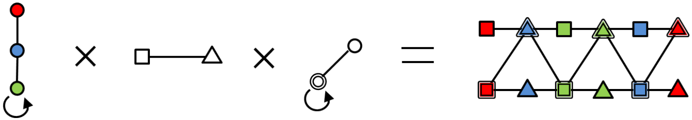
Weighted:



Directed:



Multiple Products:



Relevance and Intuition: Fractal Nature

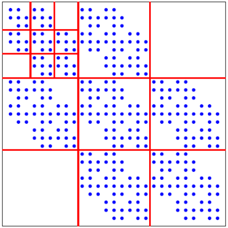
- Recursive growth of graph communities: Nodes get expanded to micro communities

1	1	0
1	1	1
0	1	1

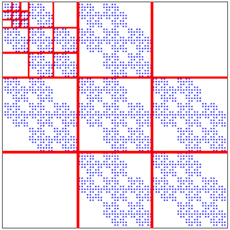
$\mathcal{A}(\mathcal{K})$

K_1	K_1	0
K_1	K_1	K_1
0	K_1	K_1

$\mathcal{A}(\mathcal{K}\times\mathcal{K})$



$\mathcal{A}(\mathcal{K}\times\mathcal{K}\times\mathcal{K})$



$\mathcal{A}(\mathcal{K}\times\mathcal{K}\times\mathcal{K}\times\mathcal{K})$

- Obey common network features: Degree distribution, density power law, diameters, spectra [Leskovec *et al.* '10]

Relevance and Intuition: Null-Models

- Attribute representations, e.g., High/Low GPA, Year in school
- Nodes describe attributes, e.g., $u = (\text{High GPA}, \text{Yr } 12)$, $v = (\text{Low GPA}, \text{Yr } 11)$
- Edges describe interaction probabilities, e.g., $p(u \to v) = 0.2 \times 0.1 = 0.02$

GPA	High	Low
High	0.8	**0.2**
Low	0.3	0.7

$\mathcal{A}(\mathcal{G}_1)$

Class	Yr12	Yr11
Yr12	0.9	**0.1**
Yr11	0.6	0.4

$\mathcal{A}(\mathcal{G}_2)$

(GPA,Class)	(High,Yr12)	(High,Yr11)	(Low,Yr12)	(Low,Yr11)
(High,Yr12)	0.72	0.08	0.18	**0.02**
(High,Yr11)	0.48	0.32	0.12	0.08
(Low,Yr12)	0.27	0.03	0.63	0.07
(Low,Yr11)	0.18	0.12	0.42	0.28

$\mathcal{A}(\mathcal{G}_1 \times \mathcal{G}_2)$

Dynamics over Kronecker Product

- The factor dynamics for $i = 1, 2, \ldots$

$$x_i(k+1) = A(\mathcal{G}_i)x_i(k)$$
$$y_i(k) = C_i x_i(k)$$

- Consider the discrete dynamics

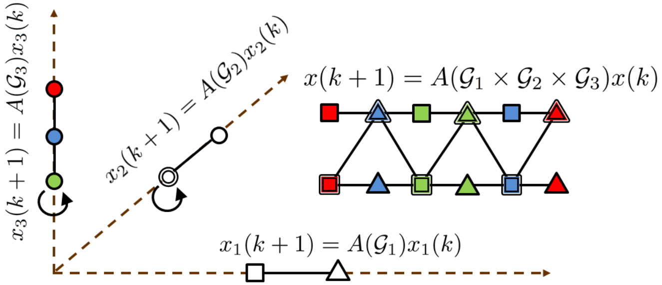
$$x(k+1) = A(\mathcal{G}_1 \times \mathcal{G}_2 \times \ldots)x(k) = A(\prod_{\times} \mathcal{G}_i)x(k)$$
$$y(k) = (C_1 \otimes C_2 \otimes \ldots)x(k) = \prod_{\otimes} C_i x(k)$$

- For output node sets $S_1, S_2, \ldots$ then $C(S_1) \otimes C(S_2) \otimes \cdots = C(S_1 \times S_2 \times \ldots)$
- Here $A(\cdot)$ preserved the Kronecker product, e.g., Adjacency $\mathcal{A}(\mathcal{G}_1)$, Row stochastic adjacency $[A_s(\mathcal{G})]_{ij} = \frac{[\mathcal{A}(\mathcal{G})]_{ij}}{\sum_j [\mathcal{A}(\mathcal{G})]_{ij}}$
- How does the features of the factors compare to the composite system?

Trajectory and Stability

When initialized from $x(0) = \prod x_i(0)$ the composite **trajectory** is

$$x(k+1) = \prod_{\otimes} x_i(k)$$

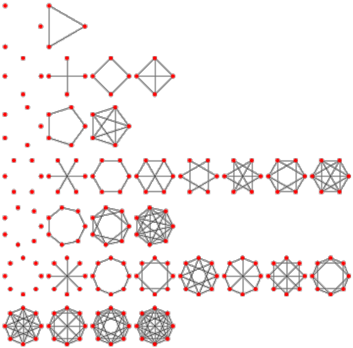


$$\boxed{x(k+1) = x_1(k) \otimes x_2(k) \otimes x_3(k)}$$

- Consequence: If the factors are stable then the composite is **stable**

Observability

- Dynamics are observable if for any unknown $x(0)$, t_f there exists a t_f such that knowledges of $u(t)$ and $y(t)$ over $[0, t_1]$ uniquely determine $x(0)$.
- Significant in networked robotic systems, human-swarm interaction, network security, quantum networks.
- Challenging to establish for large networks
- Known families of observable graphs for selected outputs
  - Paths (Rahmani and Mesbahi '07)
  - Circulants (Nabi-Abdolyousefi and Mesbahi '12)
  - Grids (Parlengeli and Notarsefano '11)
  - Distance regular graphs (Zhang *et al.* '11)
  - Cartesian products (Chapman *et al.* '14)



Observability

- Consider diagonalizable $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ with distinct eigenvalues of $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_p$ and $\tilde{\mu}_1, \ldots, \tilde{\mu}_q$

Theorem

The pair $(A(\mathcal{G}_1 \times \mathcal{G}_2), C_1 \otimes C_2)$ is observable if and only if
(1) the pairs $(A(\mathcal{G}_1), C_1)$ and $(A(\mathcal{G}_2), C_2)$ are observable and
(2) for $\tilde{\lambda}_1\tilde{\mu}_1 = \tilde{\lambda}_2\tilde{\mu}_2 = \cdots = \tilde{\lambda}_p\tilde{\mu}_p$, $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ $\forall i \neq j$, $p > 1$,

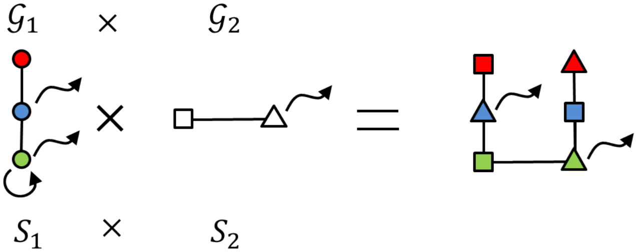
$$C_1^T \not\perp [U_1, U_2, \ldots, U_p] \text{ and/or } C_2^T \not\perp [V_1, V_2, \ldots, V_p],$$

where the columns of U_i are the orthogonal right eigenvectors of eigenvalues $\tilde{\lambda}_i$ of $A(\mathcal{G}_1)$ (sim. for pairs $(\tilde{\mu}_i, V_i)$ of $A(\mathcal{G}_2)$).

- If the factors are observable then all modes such that $\tilde{\lambda}_i\tilde{\mu}_s \neq \tilde{\lambda}_j\tilde{\mu}_t$ are **observable**
- If (1) and (2) satisfied, and C_1 and C_2 are minimal rank observable on the factors then $C_1 \otimes C_2$ is a minimal rank observable on the composite

Observability Example

- $(A(\mathcal{G}_1), C(S_1))$ is observable with $S_1 = \{\text{blue, green}\}$
- $(A(\mathcal{G}_2), C(S_2))$ is observable with $S_2 = \{\Delta\}$



- No new multiplicities are introduced in $A(\mathcal{G}_1 \times \mathcal{G}_2)$

$\implies$ $(A(\mathcal{G}_1 \times \mathcal{G}_2), C(S_1 \times S_2))$ is observable with $S_1 \times S_2 = \{\text{blue } \Delta, \text{ green } \Delta\}$

Observability Factorization - Idea of the Proof

Popov-Belevitch-Hautus (PBH) test

(A, C) is unobservable if and only if there exists a right eigenvalue-eigenvector pair (λ, v) of A such that $Cv = 0$.

- Eigenvalue and eigenvector relationship:

	$A(\mathcal{G}_1)$	$A(\mathcal{G}_2)$	$A(\mathcal{G}_1 \times \mathcal{G}_2)$
Eigenvalue	λ_i	μ_j	$\lambda_i \mu_j$
Eigenvector	v_i	u_j	$v_i \otimes u_j$

- Also $(C_1 \otimes C_2)(v_i \otimes u_i) = C_1 v_i \otimes C_2 u_i$
- The proof for simple eigenvalues follows from these observations.

Graph Factorization

- A graph can be factored as well as composed...

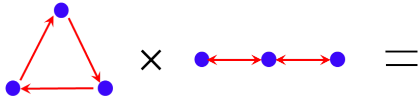
Theorem (Sabidussi 1960)

Every connected graph can be factored as a Kronecker product of prime graphs. This is NOT unique up to reordering of the factors.

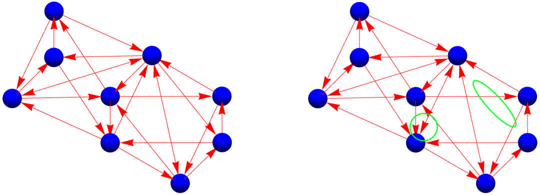
- Primes: $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ implies that either $\mathcal{G}_1$ or $\mathcal{G}_2$ is K_1
  - Number of prime factors is at most $\log |\mathcal{G}|$
- Algorithms
  - Van Loan and Pitsianis (1993) - Exact and an approximation $\mathcal{O}(n^3)$ (more later)
  - Leskovec et al. (2010) - "KronFit" approximation of the form $\mathcal{G}_1 \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_1$

Kronecker Product Approximations

- Van Loan and Pitsianis developed an efficient method to solve $\min \|A - A_1 \otimes A_2\|_{2,F}$, i.e., the closest Kronecker product $\mathcal{G} \approx \mathcal{G}_1 \times \mathcal{G}_2$



$\mathcal{G}_1$ $\mathcal{G}_2$



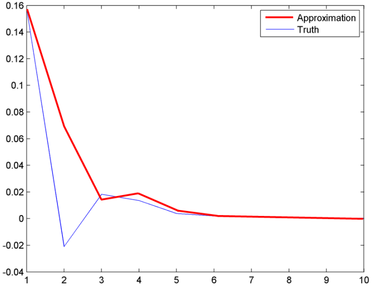
$\mathcal{G}_1 \times \mathcal{G}_2$ $\mathcal{G}$

- How does the features of these approximate factors compare to the composite system?

Trajectory Approximation

Approximate Kronecker Dynamics

$$x(k+1) = (A(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta)\,x(k)$$
$$y(k) = (C_1 \otimes C_2)\,x(k)$$



- For unforced dynamics if $x(0) = x_1(0) \otimes x_2(0)$ then the trajectory can be approximated by $x_a(k) = x_1(k) \otimes x_2(k)$ where $A_\times = A(\mathcal{G}_1 \times \mathcal{G}_2)$

$$\|x(k) - x_a(k)\| \leq \|\Delta\|\,\|x(0)\|\,\frac{\left\|A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\,I\right\|^k - \left\|A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\,I\right\|^k}{\left\|A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\,I\right\| - \left\|A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\,I\right\|}$$

Distance to Instability

- **Distance to instability** is

$$d_A = \inf(\|\Delta\| : A + \|\Delta\| \text{ is unstable}) = 1 - \rho(d_A).$$

- If the distance to instability of the factors is $d_{\mathcal{G}_1}$ and $d_{\mathcal{G}_2}$ then the distance to instability of the composite is

$$d_{\mathcal{G}_1 \times \mathcal{G}_2} = d_{\mathcal{G}_1} + d_{\mathcal{G}_2} - d_{\mathcal{G}_1} d_{\mathcal{G}_2}$$

- Consequence: A stable composite dynamics is always "more stable" than its factors dynamics, i.e.,

$$d_{\mathcal{G}_1 \times \mathcal{G}_2} \geq \max(d_{\mathcal{G}_1}, d_{\mathcal{G}_2})$$

Distance to Unobservability

- **Distance to unobservability** is

$$d_{A,C} = \inf\{\|\Delta\| : (A+\Delta, C) \text{ is unobservable }\}$$

- Let λ_1 and μ_1 are the smallest magnitude eigenvalues of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively
- If the distance to unobservability of the factors is $d_{\mathcal{G}_1,C_1}$ and $d_{\mathcal{G}_2,C_2}$ then the distance to unobservability of the composite is bounded as

$$d_{\mathcal{G}_1\times\mathcal{G}_2,C} \leq \min\left(|\lambda_1|\, d_{\mathcal{G}_2,C_2}, |\mu_1|\, d_{\mathcal{G}_1,C_1}\right)$$

- Consequence: For factor dynamics with a stable mode, the composite dynamics is always "closer" to unobservability than the factors, i.e.,

$$d_{\mathcal{G}_1\times\mathcal{G}_2,C} \leq \min\left(d_{\mathcal{G}_2,C_2}, d_{\mathcal{G}_1,C_1}\right)$$

Conclusion

- Kronecker product dynamics related to its factors examining
  - Trajectory
  - Stability
  - Observability
- Approximate Kronecker dynamics related to its approximate factors examining
  - Bounded Trajectory
  - Distance to instability
  - Distance to unobservability
- Future work:
  - Lower bounds for distance to unobservability
  - Input/Output approximations
  - Powers of Kronecker products