Technical Notes and Correspondence

System Theoretic Aspects of Influenced Consensus: Single Input Case

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Abstract—**This technical note examines the dynamics of networked multi-agent systems operating with a consensus-type algorithm, under the influence of an attached node or external agent. Depending on the specific scenario, the attached node can be viewed as a network intruder or an administrator. We introduce an influence scheme, naive of the network topology, involving predictable excitation of the network with the objective of manipulating, disrupting, or steering its evolution. The spectrum of the corresponding Dirichlet matrix provides bounds on the system-theoretic properties of the resulting influenced network, quantifying its security—or viewed differently—its manageability. Finally, the controllability gramian for influenced consensus is examined, providing insights into its $\mathcal{H}_2$-norm and controllability properties.**

Index Terms—**Consensus protocol, influence models, network management, network security.**

I. Introduction

Consensus-type algorithms provide effective means of distributed information-sharing and control for networked multi-agent systems, in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example [1]–[4]. One of the appeals of consensus algorithms is their ability to operate distributively and autonomously over simple trusting agents. This has the added benefit that external (control) agents, perceived as *native* agents, can seamlessly attach to the network and steer it in particular directions. These additional agents, ignoring consensus rules, will influence the network as compared to the *unforced* network, resulting in scenarios such as leader-follower [2], [5], and drift correction [6]. The detriment is that this same approach can be adopted by malicious infiltrating agents. We refer to consensus-based systems, with a friendly or unfriendly attached nodes, as *influenced consensus networks*. The case where such networks are influenced by one external agent is the focus of the present technical note.

Although the properties of consensus algorithms has been extensively studied, examining the input-output properties of influenced consensus is in its infancy; recent works in this direction include [7]–[9]. In the traditional unforced consensus, for example, one of the popular performance metrics is the second smallest eigenvalue of the graph Laplacian. This metric proves less attractive for influenced consensus as it fails to capture where the influencing node has attached, which in turn can dramatically influence the convergence rate of the corresponding influenced consensus.

The present technical note contributes to the general area of influenced consensus by defining performance metrics for the case where there is only one attached node. We first examine an *anchor influence* scheme where the attached node delivers a constant signal. In this venue, we demonstrate the utility of the spectrum of the modified Laplacian, referred to as the Dirichlet matrix, to parametrize the effectiveness of the influence scheme. Moreover, we present fundamental bounds on these metrics and identify structural features that characterize these metrics in terms of the number of agents (nodes) and links (edges), as well as the (graph) distances between the native agents and the influencing agent. For the scenario where the external node is malicious, these structural connections can be exploited to improve network security. An illustrative example is provided to such an effect.

Secondly, we consider disrupting the network via more arbitrary signals, enabling us to analyze the manageability of the network. Using the influenced system's *controllability gramian*, we show that the $\mathcal{H}_2$-norm of the corresponding input-output system is independent of the network structure. Moreover, we refine the classification of network controllability by showing the importance of the gramian in practical controllability of graphs.

The current work is part of a more general effort that aims to identify bounds and metrics on the security and manageability of coordination algorithms via the network structure when influenced by an external friendly or unfriendly agent. As such, our work is related to a number of other research works, such as those in computer network security [10], spread of epidemics [11], [12], predator/prey swarming [13], and controlled or intruded consensus protocols [7], [14], [15].

A. Background, Notation, and the Influence Model

In this section, we provide the background on constructs that will be used subsequently in the technical note, including an abbreviated description of graphs and the consensus protocol. The reader is referred to [16] for a more detailed exposition on graph theory, particularly, algebraic graph theory; for the consensus protocol, see the survey [1]. An undirected graph $\mathcal{G} = (V, E)$ is defined by a vertex (or node) set $V = \{v_1, v_2, \cdots, v_n\}$ and an edge set $E \subseteq [V]^2$ of cardinality m.[1] [2] In this technical note, we will refer to node v_i as agent i. The nodes v_i and v_j are called adjacent if $\{v_i, v_j\} \in E$. The degree $d_i(\mathcal{G})$ of node $v_i \in V$ is the number of its adjacent nodes. The minimum and maximum degrees in the graph $\mathcal{G}$ will be denoted by $d_{\min}(\mathcal{G})$ and $d_{\max}(\mathcal{G})$, respectively. The degree matrix $\Delta(\mathcal{G})$ is a diagonal matrix with $d_i(\mathcal{G})$ on the ith diagonal entry. The adjacency matrix, on the other hand, is a symmetric matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $\{v_i, v_j\} \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The combinatorial Laplacian of the graph, defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, is a (symmetric) positive semi-definite matrix. The Laplacian spectrum is assumed to be ordered as $0 = \lambda_1(\mathcal{G}) \le \lambda_2(\mathcal{G}) \le \ldots \le \lambda_n(\mathcal{G})$; for brevity we will continue to use $\lambda_i(\mathcal{G})$ instead of $\lambda_i(L(\mathcal{G}))$. This ordering convention for eigenvalues will also be used for other symmetric matrices. The notation $A \ge B$ for two symmetric matrices signifies the positive semi-definiteness of the difference $A - B$. When this is the case, one has $\lambda_i(A) \ge \lambda_i(B)$ for all i, a fact that will be subsequently used. Our analysis will involve the spectrum of the Laplacian and its *influenced* version, which is referred to as the Dirichlet matrix due to its

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[1]The notation $[V]^2$ refers to the set of two-element subsets of V.

[2]Some special n-node graphs are the path graph $\mathcal{P}$, star graph $\mathcal{S}$, and complete graph $\mathcal{K}$. These can be defined via their edge sets as $E = \{\{j-1, j\} | j \in \{2, \ldots, n\}\}$, $E = \{\{1, j\} | j \in \{2, \ldots, n\}\}$, and $E = \{\{i, j\} | i \neq j, i, j \in \{1, \ldots, n\}\}$, respectively.

resemblance with the Dirichlet operator encountered in PDE models with boundaries [17], [18].

Viewing each node as a one-dimensional single integrator, let $x_i(t) \in \mathbb{R}$ denote the state of node $v_i \in V$ at time t. The continuous-time consensus protocol for agent i is then defined as $\dot{x}_i(t) = \sum_{\{i,j\}\in E}(x_j(t) - x_i(t))$, which in its compact form with $x(t) \in \mathbb{R}^n$ is written as

$$\dot{x}(t) = -L(\mathcal{G})x(t). \tag{1}$$

From the definition of the graph Laplacian all rows of $L(\mathcal{G})$ sum to zero and $\lambda_1(\mathcal{G}) = 0$ with the corresponding eigenvector $\mathbf{1} = [1,\ldots,1]^T \in \mathbb{R}^n$. Subsequently, when $\mathcal{G}$ is connected, it can be deduced that the consensus protocol steers all agents to the average value of their initial states [1].

The model that we will examine in this note is the *single input influenced consensus*. This model can be formalized by considering an influencing node attached to a native node in the graph $v_i \in V$, delivering a signal $u(t) \in \mathbb{R}$. The resulting linear time-invariant model assumes the form

$$\dot{x}(t) = A(\mathcal{G},i)x(t) + B(i)u(t) \tag{2}$$

where $B(i) = e_i \in \mathbb{R}^n$ with $[e_i]_j = 1$ for $j = i$ and $[e_i]_j = 0$ otherwise, and $A(\mathcal{G},i) = -(L(\mathcal{G}) + e_i e_i^T)$.

The matrix $-A(\mathcal{G},i)$, which is referred to as the Dirichlet matrix, can be formed by adding "1" to the ith diagonal entry of $L(\mathcal{G})$ in (1). Consequently, by closely relating the spectra of $L(\mathcal{G})$ and $A(\mathcal{G},i)$, features of the underlying graph $\mathcal{G}$ can be related to model (2). The following result highlights this connection.

Proposition 1.1: The eigenvalues of the matrix $-A(\mathcal{G},i)$ in (2) satisfy the following inequalities:

(a) $\lambda_j(\mathcal{G})) \leq \lambda_j(-A(\mathcal{G},i))$;

(b) $\lambda_j(-A(\mathcal{G},i)) \leq \lambda_j(\mathcal{G}) + 1$;

(c) for $j > 1$, $\lambda_{j-1}(-A(\mathcal{G},i)) \leq \lambda_j(\mathcal{G}))$.

Proof: The matrix $-A(\mathcal{G},i)$ is the sum of two positive semidefinite matrices $L(\mathcal{G})$ and $e_i e_i^T$. As such, the matrix $-A(\mathcal{G},i)$ is positive semidefinite. By the eigenvalue interlacing theorem ([19], Corollary 4.3.3 and Theorem 4.3.6), bounds (a) and (c) follow. Moreover, Weyl's Theorem ([19], Theorem 4.3.1) implies that

$$\begin{aligned}\lambda_j(-A(\mathcal{G},i)) &= \lambda_j\left(L(\mathcal{G}) + e_i e_i^T\right)\\ &\leq \lambda_j(\mathcal{G}) + \lambda_n\left(e_i e_i^T\right)\\ &= \lambda_j(\mathcal{G}) + 1.\end{aligned}$$

■

The rest of the technical note is organized as follows. We first examine the *anchor influence* scheme in Section II where the external node injects a constant signal to the network. In this setting, we provide bounds on a quadratic performance metric to characterize how secure or manageable the network may be. Next, gathering insights into arbitrary influence schemes, the *controllability gramian* of the graph is examined in Section III. The Appendix details the performance costs and the controllability gramian for special types of graphs. We conclude the technical note with a few remarks in Section IV.

II. Anchor Influence

The anchor influence scheme adopts a naive approach for influencing the network, justified by the lack of a prior knowledge of the network structure by the attached node. In this case, the attached node merely attempts to steer the system to a common state u_c.[3] This section is devoted to characterizing how effective such an influence scheme can be as a function of the graph structure and where the external node has been attached. We note that when the network is driven by a stochastic signal with a constant expected value $\mathbf{E}\{u\} = u_c$, the evolution of the expected value of the nodes' state $\mathbf{E}\{x\}$ can be modeled as (2) where $u(t) = u_c$ for all time t.

Before continuing our discussion, let us state an auxiliary result that is subsequently used; for the proof see for example [15].

Proposition 2.1: When the original graph $\mathcal{G}$ is connected, the matrix $A(\mathcal{G},i)$ in model (2) is negative definite.

We note that a consequence of Proposition 2.1 is that the subspace spanned by $\mathbf{1}$ is reachable for the influenced system (2), when $\mathcal{G}$ is connected. In fact, in the case where $u(t) = u_c$ in model (2), all agents' state converge to u_c. We now examine the corresponding *state cost* for the injected signal u_c by the attached node over an infinite horizon, in order to steer the network to the consensus state $u_c\mathbf{1}$ from an arbitrary initialization. More specifically, noting that $A(\mathcal{G},i)^{-1}B(i) = -\mathbf{1}$, the convergence cost over an infinite time horizon with $\tilde{x}(t) = x(t) - u_c\mathbf{1}$ can be derived as[4]

$$\begin{aligned}J(\mathcal{G},i,\tilde{x}(0)) &= 2\int_0^\infty \tilde{x}^T\tilde{x}\,dt\\ &= \int_0^\infty \tilde{x}^T x + x^T\tilde{x} - \tilde{x}^T\mathbf{1}u_c - u_c\mathbf{1}^T\tilde{x}\,dt\\ &= \int_0^\infty \tilde{x}^T x + x^T\tilde{x} + \tilde{x}^T A(\mathcal{G},i)^{-1}B(i)u_c\\ &\quad + u_c B(i)^T A(\mathcal{G},i)^{-1}\tilde{x}\,dt\\ &= \int_0^\infty (A(\mathcal{G},i)x + B(i)u_c)^T A(\mathcal{G},i)^{-1}\tilde{x}\\ &\quad + \tilde{x}^T A(\mathcal{G},i)^{-1}(A(\mathcal{G},i)\tilde{x} + B(i)u_c)\,dt\\ &= \int_0^\infty \dot{\tilde{x}}^T A(\mathcal{G},i)^{-1}\tilde{x} + \tilde{x}^T A(\mathcal{G},i)^{-1}\dot{\tilde{x}}\,dt\\ &= \int_0^\infty \left\{\frac{d}{dt}\tilde{x}^T A^{-1}\tilde{x}\right\}dt\\ &= \tilde{x}(\infty)^T A(\mathcal{G},i)^{-1}\tilde{x}(\infty) - \tilde{x}(0)^T A(\mathcal{G},i)^{-1}\tilde{x}(0)\\ &= -\tilde{x}(0)^T A(\mathcal{G},i)^{-1}\tilde{x}(0).\end{aligned} \tag{3}$$

In view of (3), we parametrize the resilience of a consensus-type network to anchor influence.[5] We define three metrics, referred to as the minimum, maximum, and average performance costs in terms of the eigenvalues of the Dirichlet matrix as[6]

$$\begin{aligned}J^{\min}(\mathcal{G},i) &= \inf_{\|\tilde{x}(0)\|=1} J(\mathcal{G},i,\tilde{x}(0))\\ &= \lambda_1\left(-A(\mathcal{G},i)^{-1}\right)\\ &= 1/\lambda_n(-A(\mathcal{G},i))\end{aligned} \tag{4}$$

$$\begin{aligned}J^{\max}(\mathcal{G},i) &= \sup_{\|\tilde{x}(0)\|=1} J(\mathcal{G},i,\tilde{x}(0))\\ &= \lambda_n\left(-A(\mathcal{G},i)^{-1}\right)\\ &= 1/\lambda_1(-A(\mathcal{G},i))\end{aligned} \tag{5}$$

[3]For example in order to realign a formation or to change its speed.

[4]The scaling by 2 is mostly cosmetic. Further, $\tilde{x}(t)$ is abbreviated as $\tilde{x}$ for brevity.

[5]In our subsequent discussion, we will only consider connected graphs, as each component of disconnected graphs can be analyzed separately.

[6]Considered as intrusion or management costs. The expectation in (6) is with respect to the uniform distribution on the unit ball.

and

$$\begin{aligned} J^{\mathrm{avg}}(\mathcal{G},i) &= \mathbf{E}_{\|\tilde{x}(0)\|=1} J\left(\mathcal{G},i,\tilde{x}(0)\right) \\ &= \frac{1}{n}\mathbf{tr}\left(-A(\mathcal{G},i)^{-1}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} 1/\lambda_i\left(-A(\mathcal{G},i)\right) \end{aligned} \tag{6}$$

where $\mathbf{tr}$ is the matrix trace operator. As we will show, the critical graphs that bound these performance costs for a single input anchor influence scheme over all graphs are the n-node complete graph $\mathcal{K}$, the path graph $\mathcal{P}$, and the star graph $\mathcal{S}$. The derivation of the corresponding metrics relating to these graphs are relegated to the Appendix.

Proposition 2.2: For an n-node connected graph $\mathcal{G}$, the minimum and maximum performance costs (4), (5) of attaching to a node $v_i \in \mathcal{G}$ is bounded as

$$\inf_{(\mathcal{G},i)} J^{\min}(\mathcal{G},i) = 2(1+n+\sqrt{n^2+2n-3})^{-1} \tag{7}$$

and

$$\sup_{(\mathcal{G},i)} J^{\max}(\mathcal{G},i) = \frac{1}{2}\left(1+\cos\frac{2\pi n}{2n+1}\right)^{-1}. \tag{8}$$

Proof: Let $\mathcal{G}$ be an arbitrary n-node graph with its complement graph $\widehat{\mathcal{G}}$, noting that $L(\mathcal{G}) + L(\widehat{\mathcal{G}}) = L(\mathcal{K})$, where $\mathcal{K}$ is the n-node complete graph. Since

$$L(\mathcal{G}) + L(\widehat{\mathcal{G}}) + e_ie_i^T = L(\mathcal{K}) + e_ie_i^T \tag{9}$$

it follows that:

$$\lambda_n\left(L(\mathcal{G}) + e_ie_i^T\right) \le \lambda_n\left(L(\mathcal{K}) + e_ie_i^T\right). \tag{10}$$

Therefore

$$\begin{aligned} J^{\min}(\mathcal{K},i) &= \left(\lambda_n\left(L(\mathcal{K}) + e_ie_i^T\right)\right)^{-1} \\ &\le \left(\lambda_n\left(L(\mathcal{G}) + e_ie_i^T\right)\right)^{-1} \\ &= J^{\min}(\mathcal{G},i). \end{aligned} \tag{11}$$

Now consider a spanning tree $\mathcal{T}$ of the graph $\mathcal{G}$. Attaching an influencing node to $\mathcal{T}$ and examining the corresponding smallest Dirichlet eigenvalue, we obtain

$$\lambda_1\left(L(\mathcal{T}) + e_ie_i^T\right) \le \lambda_1\left(L(\mathcal{G}) + e_ie_i^T\right). \tag{12}$$

Next, construct the new tree $\widetilde{\mathcal{T}}$ by mirroring $\mathcal{T}$ about the influencing node and treating it as a *native* node in this new graph which has $2n+1$ nodes. From Lemma 6 of [20], it follows that:

$$\lambda_2(\widetilde{\mathcal{T}}) \le \lambda_1\left(L(\mathcal{T}) + e_ie_i^T\right). \tag{13}$$

In the meantime, as the path graph is the tree with the least second smallest eigenvalue over all n-node connected graphs [21], it follows that $\lambda_2(\widetilde{\mathcal{P}}) \le \lambda_2(\widetilde{\mathcal{T}})$, where $\widetilde{\mathcal{P}}$ is a path of order $2n+1$. From Proposition A.1 of the Appendix, for a path graph $\mathcal{P}$ of order n, with the influence node attached to $v_1 \in \mathcal{P}$, it is known that $\lambda_2(\widetilde{\mathcal{P}}) = \lambda_1(-A(\mathcal{P},1))$; see also [21]. Combining these bounds, we arrive at the inequality

$$\lambda_2(\widetilde{\mathcal{P}}) \le \lambda_1\left(L(\mathcal{T}) + e_ie_i^T\right) \tag{14}$$

and thereby

$$\begin{aligned} J^{\min}(\mathcal{G},\mathcal{R}^i) &= \frac{1}{\lambda_1\left(L(\mathcal{G}) + e_ie_i^T\right)} \\ &\le \frac{1}{\lambda_1\left(-A(\mathcal{P},1)\right)} \\ &= J^{\min}(\mathcal{P},1). \end{aligned} \tag{15}$$

Closed form solutions for $J^{\min}(\mathcal{K},i)$ and $J^{\min}(\mathcal{P},1)$ are found in Propositions A.1 and A.5 of the Appendix, thus completing the proof. ■

There is an intuitive link between the "centrality" of a node in a network and its influence on the network's dynamics as an intruder or administrator. This correlation becomes apparent for tree graphs in relation to the average performance cost (6) as we proceed to show.

Lemma 2.3: For an n-node tree $\mathcal{T}$, the average performance cost (6) of attaching to node $v_i \in \mathcal{T}$ is

$$J^{\mathrm{avg}}(\mathcal{T},i) = \frac{1}{n}\sum_{j=1}^{n} d(v_i,v_j) + 1 \tag{16}$$

where $d(v_i,v_j)$ denotes the length of the shortest path between nodes v_i and v_j.

Proof: Without loss of generality, assume that the influencing node is attached to $v_1 \in \mathcal{T}$. The inverse of the system matrix thus assumes the form

$$-A(\mathcal{G},1)^{-1} = \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \tag{17}$$

where $Z = A_{11}^{-1} + \mathbf{1}\mathbf{1}^T$ and A_{11} is the principal submatrix of $-A(\mathcal{G},1)$ formed from deleting its first row and column. We will consider the cases where v_1 is adjacent to a single node or multiple nodes within the graph separately in order to characterize the diagonal elements of $-A(\mathcal{G},1)^{-1}$.

Case 1) If node v_1 is adjacent to a single node, say v_2, then $A_{11} = -A(\mathcal{G}_{n-1},1)$, where $\mathcal{G}_{n-1}$ is the subgraph of $\mathcal{G}$ formed by removing node v_1 and attaching the influence node to node v_2.

Case 2) If node v_1 is adjacent to more than one node, say $v_2,\ldots,v_p$, then as the original graph is a tree, the corresponding A_{11} would represent $p-1$ tree graphs $\mathcal{G}_2,\ldots,\mathcal{G}_p$, each with an influence node attached to nodes $v_2,\ldots,v_p$. In fact, with a possible node relabeling

$$A_{11} = -\begin{bmatrix} A(\mathcal{G}_2,1) & 0 & \cdots & 0 \\ 0 & A(\mathcal{G}_3,1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A(\mathcal{G}_p,1) \end{bmatrix}. \tag{18}$$

The two cases can now be reapplied, one link at a time, with each subgraph decreasing the dimension of its respective submatrices by 1. This process could be continued until we have a set of submatrices of the form $-A(\mathcal{G}_q,1) = 1$. In the meantime, this procedure confirms that the diagonal elements of $-A(\mathcal{G},1)^{-1}$ represent the minimum distances between node v_1 and other nodes in the graph plus one. This observation, in conjunction with (6), now completes the proof. ■

Corollary 2.4: For an n-node tree $\mathcal{T}$, the average performance cost (6) of attaching to an arbitrary node $v_i \in \mathcal{T}$ is bounded as

$$\inf_{(\mathcal{T},i)} J^{\mathrm{avg}}(\mathcal{T},i) = 2 - \frac{1}{n} \tag{19}$$

and

$$\sup_{(\mathcal{T},i)} J^{\mathrm{avg}}(\mathcal{T},i) = \frac{1}{2}(n+1). \tag{20}$$

Proof: Over all n-node connected trees, the central node of the star graph has the smallest minimum distance of 1 to all other nodes and an end node of the path graph has the largest accumulative distance of $\sum_{i=1}^{n-1} i$ to all other nodes. These observations, in conjunction with Lemma 2.3, complete the proof. ■

Proposition 2.5: For a graph $\mathcal{G}$ with n-nodes and m-edges with largest node degree $d_{\max}(\mathcal{G})$ and smallest node degree $d_{\min}(\mathcal{G})$, the minimum, maximum, and average performance costs (4)–(6) of attaching to any node $v_i \in \mathcal{G}$ are bounded as

$$\frac{1}{2d_{\max}(\mathcal{G})+1} \le J^{\min}(\mathcal{G}, i) \tag{21}$$

$$J^{\max}(\mathcal{G}, i) \le \frac{1}{2}(n^2+3n-2m-2) \tag{22}$$

$$\frac{n}{4m} - \frac{1}{2nd_{\min}(\mathcal{G})\,(2d_{\min}(\mathcal{G})+1)} \le J^{\text{avg}}(\mathcal{G}, i) \tag{23}$$

and

$$J^{\text{avg}}(\mathcal{G}, i) \le \frac{1}{2n}(n^2 + 3n - 2m - 2). \tag{24}$$

Proof: Let the vertices in $\mathcal{G}$ be labelled such that the diagonal entries of the matrix $2\Delta(\mathcal{G}) + e_i e_i^T$ are ordered in a non-decreasing order; thus $\lambda_j((-A(\mathcal{G}, i))) \le 2d_j(\mathcal{G})$ for all j. However, since

$$\lambda_n\left(-A(\mathcal{G}, i)\right) \le \lambda_n\left(-A(\mathcal{G}, n)\right) \le 2d_{\max}(\mathcal{G}) + 1 \tag{25}$$

it follows that:

$$(2d_{\max}(\mathcal{G}) + 1)^{-1} \le \lambda_n\left(-A(\mathcal{G}, i)\right)^{-1} = J^{\min}(\mathcal{G}, i). \tag{26}$$

Similarly

$$\begin{aligned} nJ^{\text{avg}}(\mathcal{G}, i) &= \sum_{j=1}^{n} \lambda_j\left(-A(\mathcal{G}, i)\right)^{-1} \\ &\ge \left(\sum_{j=1}^{n} \frac{1}{2d_j(\mathcal{G})}\right) - \frac{1}{2d_{\min}(\mathcal{G})} + \frac{1}{2d_{\min}(\mathcal{G}) + 1} \\ &\ge \frac{1}{2}\left(\sum_{j=1}^{n} \frac{n}{2m} - \frac{1}{d_{\min}(\mathcal{G})\,(2d_{\min}(\mathcal{G}) + 1)}\right) \\ &\ge \frac{n^2}{4m} - \frac{1}{2d_{\min}(\mathcal{G})\,(2d_{\min}(\mathcal{G}) + 1)}. \end{aligned} \tag{27}$$

Consider next a spanning tree $\mathcal{T}$ of $\mathcal{G}$. Then $0 \preceq L(\mathcal{G}) - L(\mathcal{T}) = L(\mathcal{G}) + e_i e_i^T - (L(\mathcal{T}) + e_i e_i^T) = -A(\mathcal{G}, i) - (-A(\mathcal{T}, i))$ and thereby $\lambda_j(-A(\mathcal{T}, i)) \le \lambda_j(-A(\mathcal{G}, i))$, for $j = 1, \ldots, n$. Consequently

$$\begin{aligned} nJ^{\text{avg}}(\mathcal{G}, i) &\le \sum_{j=1}^{n} \lambda_j\left(-A(\mathcal{T}, i)\right)^{-1} \\ &\le \sup_{\mathcal{T} \subseteq \mathcal{G}} nJ^{\text{avg}}(\mathcal{T}, i) \\ &= \frac{1}{2}(n^2 + 3n - 2m - 2) \end{aligned} \tag{28}$$

where the last inequality is from examining spanning trees of all n-nodes m-edges graphs, which in turn provides the final equality from $\sum_{j=1}^{n} d(v_i, v_j) \le (1/2)(n-1)(n+2) - m$ [17] and Lemma 2.3. Reapplying this bound we also have

$$\begin{aligned} J^{\max}(\mathcal{G}, i) &= \lambda_1\left(-A(\mathcal{G}, i)\right)^{-1} \\ &\le \sum_{j=1}^{n} \lambda_j\left(-A(\mathcal{G}, i)\right)^{-1} \\ &= nJ^{\text{avg}}(\mathcal{G}, i) \\ &\le \frac{1}{2}(n^2 + 3n - 2m - 2) \end{aligned} \tag{29}$$

thus completing the proof. ■

A. An Example

The average performance cost (6) can be employed to design a protocol over a tree $\mathcal{T}$ to locally trade edges between adjacent nodes with the objective of decreasing the influence of a node attached to the network and feeding in a constant mean noise. As we have shown in Lemma 2.3, when an influencing node is attached to node $v_1 \in \mathcal{T}$, then $J^{\text{avg}}(\mathcal{T}, 1)$ is proportional to $\sum_{i=2}^{n} d(v_1, v_i)$. We now consider a scenario where v_1 broadcasts a distress signal, informing the network that it is being influenced by an external agent. Subsequently, after at most $n - 2$ rebroadcasts, all nodes in the graph are aware of the local direction of the intruding node, and more specifically, their adjacent node that is closest to v_1; we denote this node by $\mathcal{I}(v_i)$. The following local rules, requiring node v_i only knowledge of $\mathcal{I}(v_i)$ and the ability to instruct its neighbors to form edges, detailed below in Protocol 1, can then be executed asynchronously in an arbitrary order, guaranteeing that $\sum_{i=2}^{n} d(v_1, v_i)$, and hence $J^{\text{avg}}(\mathcal{T}, 1)$, increases at each iteration, and a connected tree is maintained. In fact the proposed protocol ensures that the interconnection between the host nodes eventually reaches a configuration with greatest

$$J^{\text{avg}}(\mathcal{T}, 1) = \frac{1}{2}(n + 1) \tag{30}$$

(refer to Corollary 2.4) corresponding to a path graph with the influencing node attached at one of its ends.

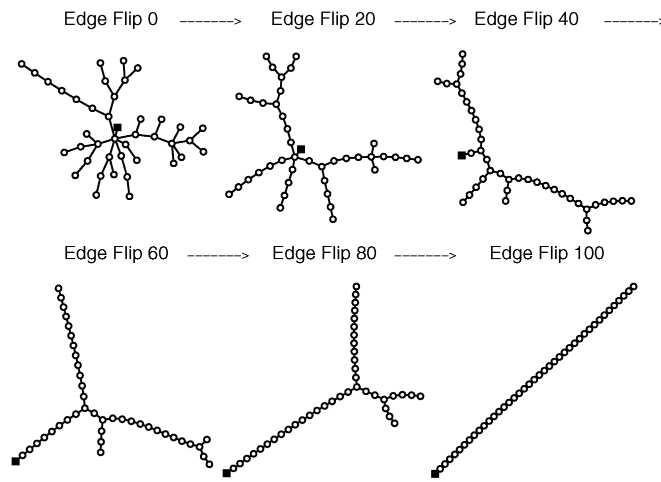

Fig. 1. Sample configurations achieved by applying Protocol 1 to a random tree. The filled square represents the influencing node.

Protocol 1 Edge swap protocol

for all Node v_i **do**

  if $\exists v_j, v_k$ where $\{v_i, v_j\}, \{v_i, v_k\} \in E$, $v_j \neq v_k$ and $v_j, v_k \notin \mathcal{I}(v_i)$ **then**

    $E \rightarrow E \setminus \{v_i, v_j\} \bigcup \{v_j, v_k\}$

  end if

end for

Fig. 1 depicts some of the intermediate graphs obtained as the protocol is applied to a random tree graph on 40 nodes. The path graph with the influencing node attached to one of its end nodes is achieved after 100 local edge flips.

III. Controllability Gramian

Up to now, our analysis hinged on the assumption that the attached node injects a constant (mean) signal into the network. Motivated by the situation where the external agent might inject an arbitrary signal to the network, we consider the controllability gramian for model (2), defined as

$$P(\mathcal{G},i) := \int_0^\infty e^{A(\mathcal{G},i)\tau} B(i)B(i)^T e^{A(\mathcal{G},i)^T\tau} d\tau \tag{31}$$

which can also be obtained by the solution to the Lyapunov equation

$$A(\mathcal{G},i)P(\mathcal{G},i) + P(\mathcal{G},i)A(\mathcal{G},i)^T = -B(i)B(i)^T. \tag{32}$$

It is well-known that the controllability gramian has a number of system theoretic properties, such as, parameterizing the system's $\mathcal{H}_2$ norm, which in turn, can measure the energy of the response of the influenced consensus dynamics to zero-mean Gaussian with unit covariance. Moreover, the gramian of the influenced consensus can characterize the energy optimal controller for steering the state of the network from one state to the another when the network is controllable. Yet another facet of the importance of the gramian pertains to its spectral properties, which are used to provide a more refined characterization of the notion of controllability. For example, when the influenced consensus dynamics (2) is controllable, $\lambda_1(P(\mathcal{G},i))$ and $\lambda_n(P(\mathcal{G},i))$ distinguish directions specified by the corresponding eigenvectors that are least and most responsive to the input with unit energy.

We first focus on $\mathbf{tr}(P(\mathcal{G},i))$ quantifies the $\mathcal{H}_2$-norm of the externally influenced consensus network.

Lemma 3.1: For all connected graphs $\mathcal{G}$ and arbitrary $v_i \in \mathcal{G}$, $\mathbf{tr}(P(\mathcal{G},i)) = 1/2$.

Proof: We note that

$$\begin{aligned}\mathbf{tr}\,(P(\mathcal{G},i)) &= \mathbf{tr}\left(\int_0^\infty e^{A(\mathcal{G},i)\tau} Q e^{A(\mathcal{G},i)^T\tau} d\tau\right)\\ &= \mathbf{tr}\left(Q\int_0^\infty e^{2A(\mathcal{G},i)\tau} d\tau\right)\\ &= -\frac{1}{2}\mathbf{tr}\left(QA(\mathcal{G},i)^{-1}\right)\end{aligned} \tag{33}$$

where $Q = e_i e_i^T$. Since $A(\mathcal{G},i)^{-1} = -\begin{bmatrix}1 & \mathbf{1}^T\\ \mathbf{1} & Z\end{bmatrix}$ (refer to the proof of Lemma 2.3), with $Z \in \mathbb{R}^{(n-1)\times(n-1)}$, it follows that $\mathbf{tr}(QA(\mathcal{G},i)^{-1}) = -1$ and $\mathbf{tr}(P(\mathcal{G},i)) = 1/2$. ∎

The implication of Lemma 3.1 is that an external agent has the same effect on the $\mathcal{H}_2$-norm of the influenced network regardless of the structure of the network and where the influenced node is attached.

An influenced consensus network with an external agent attached to the end node of a path graph has previously been established as "controllable" in [7]. The following proposition illustrates that there exists weakly controllable eigenvalues of $P(\mathcal{P},1)$ (with an end node of the path graph labeled as v_1). In fact the smallest eigenvalue of the controllability gramian $P(\mathcal{P},1)$ approaches zero as the order of the graph increases.

Proposition 3.2: For $n \geq 4$, the n-node path graph $\mathcal{P}$ with an influencing node attached to the end node $v_1 \in \mathcal{P}$, the smallest eigenvalue of the controllability gramian is of order $(2/n)^7$.

Proof: Consider the spectral decomposition $A(\mathcal{P},1) = V\Lambda V^T$, where Λ is a diagonal matrix of eigenvalues in increasing order and set $\widetilde{P} = V^T P(\mathcal{P},1)V$. From Proposition A.2 and the proof of Proposition A.1, it follows that:

$$[\widetilde{P}]_{km} = \frac{2}{2n+1}\frac{\sin k\beta \sin m\beta}{2+\cos k\beta + \cos m\beta} \tag{34}$$

where $\beta = 2\pi/(2n+1)$. Note that $P(\mathcal{P},1)$ and $\widetilde{P}$ share the same set of eigenvalues. Denote by $\widetilde{P}_{22}$ the 2×2 leading principal submatrix of $\widetilde{P}$, i.e.

$$\widetilde{P}_{22} = \frac{\beta}{\pi}\begin{bmatrix}\frac{\sin^2\beta}{2+2\cos\beta} & \frac{\sin\beta\sin 2\beta}{2+\cos\beta+\cos 2\beta}\\ \frac{\sin\beta\sin 2\beta}{2+\cos\beta+\cos 2\beta} & \frac{\sin^2 2\beta}{2+2\cos 2\beta}\end{bmatrix}. \tag{35}$$

Let $\lambda_1(\widetilde{P}_{22})$ be the smallest eigenvalue of $\widetilde{P}_{22}$ which is a differentiable function of β. Let

$$f(\beta) = \frac{\partial^7}{\partial\beta^7}\lambda_1(\widetilde{P}_{22}) \tag{36}$$

and note that $f(\beta)$ is positive and decreasing for all $n \geq 5$ (i.e., the interval $0 \leq \beta \leq (2\pi/11)$). Now by bounding the sixth order Taylor expansion about $\beta = 0$, we obtain

$$\begin{aligned}\lambda_1(\widetilde{P}_{22}) &\leq \left|f\left(\frac{2\pi}{11}\right)\right|\frac{1}{7!}\left(\frac{2\pi}{2n+1}\right)^7\\ &= \frac{2^{14}}{(2n+1)^7}\\ &< \frac{2^{14}}{(2n)^7} = \left(\frac{2}{n}\right)^7\end{aligned} \tag{37}$$

where we have used the inequality $f(2\pi/11) \leq 7!(2/\pi)^7$. By the eigenvalue interlacing theorem ([19], Theorem 4.3.15), it now follows that $\lambda_1(\widetilde{P}) \leq \lambda_1(\widetilde{P}_{22})$. For the case where $n=4$, the smallest eigenvalue of the path graph gramian can be checked directly to satisfy $\lambda_1(P(\mathcal{P},1) < (2/4)^7$. ∎

A graph feature that leads to uncontrollable influenced consensus is identified through the following proposition.

Proposition 3.3: Consider an n-node graph $\mathcal{G}$ with an influencing node attached to node $v_i \in \mathcal{G}$ such that there is an edge between v_i and all other nodes. Then

$$P(\mathcal{G},i) = \frac{1}{2(n+1)}\left(\mathbf{1}\mathbf{1}^T + e_i e_i^T\right). \tag{38}$$

In this case, there is exactly two controllable modes.

Proof: When the condition of the proposition holds, the graph $\mathcal{G}$ must have an n-node star subgraph $\mathcal{S}$ with v_i as its center node. The remaining edges in the graph form a subgraph $\widetilde{\mathcal{G}}$ which is disconnected from node v_i and thus $L(\widetilde{\mathcal{G}})e_i = 0$ which implies that

$$L(\widetilde{\mathcal{G}})P(\mathcal{G},i) + P(\mathcal{G},i)L(\widetilde{\mathcal{G}})^T = 0. \tag{39}$$

Moreover, we can rewrite the state matrix as

$$A(\mathcal{G},i) = A(\mathcal{S},i) - L(\widetilde{\mathcal{G}}). \tag{40}$$

From Proposition A.4

$$A(\mathcal{S},i)P(\mathcal{G},i) + P(\mathcal{G},i)A(\mathcal{S},i)^T = -B(i)B(i)^T \tag{41}$$

and thereby the gramian satisfies the Lyapunov equation for the influenced graph $\mathcal{G}$. ∎

The eigenvalues and eigenvectors of $P(\mathcal{G},i)$ (38) are stated in Proposition A.4. From a security perspective, the above proposition has the following ramification: *an access point to a consensus network located at nodes that are connected to all other nodes will allow at most two modes to be controllable*.

IV. Conclusion

This technical note examines consensus-type coordination networks influenced by an external node. The spectra of the Dirichlet matrix has been used to form analytic bounds on the cost of influencing such networks with a constant (mean) signal. The controllability gramian is used subsequently to shed light on the controllability and the $\mathcal{H}_2$-norm of the influence model.

APPENDIX

The following are performance costs and controllability gramians for the path graph $\mathcal{P}$, the star graph $\mathcal{S}$, and the complete graph $\mathcal{K}$, under the influence of an attached node.

Proposition A.1: For the n-node path graph $\mathcal{P}$, with an influencing node attached to an end node $v_1 \in \mathcal{P}$, the minimum, maximum, and average performance costs (4)–(6) are, respectively

$$J^{\min}(\mathcal{P},1) = \frac{1}{2}\left(1+\cos\frac{2\pi}{2n+1}\right)^{-1} \tag{42}$$
$$J^{\max}(\mathcal{P},1) = \frac{1}{2}\left(1+\cos\frac{2\pi n}{2n+1}\right)^{-1} \tag{43}$$

and

$$J^{\text{avg}}(\mathcal{P},1) = \frac{1}{2}(n+1). \tag{44}$$

Proof: The Dirichlet matrix for an influencing agent attached to v_1, an end node of $\mathcal{P}$, is

$$-A(\mathcal{P},1) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \tag{45}$$

with

$$\lambda_i\left(-A(\mathcal{P},1)\right) = 2\left(1+\cos\frac{2\pi i}{2n+1}\right) \tag{46}$$

for $i=1,\ldots,n$, with the corresponding eigenvector with its jth entry $\sin(2\pi i j/2n+1)$; see [22]. By symmetry, the scenario where the other end of the path is influenced is identical. ■

Proposition A.2: For the n-node path graph $\mathcal{P}$, with influence node attached to an end node $v_1 \in \mathcal{P}$, the controllability gramian for (2), defined element-wise for each entry (p,q), is

$$[P(\mathcal{P},1)]_{pq} = \frac{8}{(1+2n)^2}\sum_{w=1}^{n}\sum_{z=1}^{n}\frac{\sin pw\beta \sin qz\beta \sin w\beta \sin z\beta}{2+\cos w\beta + \cos z\beta} \tag{47}$$

where $\beta = 2\pi/(2n+1)$.

Proof: The proof follows from verifying that (47) satisfies the Lyapunov equation. ■

Proposition A.3: For the n-node star graph $\mathcal{S}$, with the influence node attached to the central node $v_1 \in \mathcal{S}$, the minimum, maximum, and average performance costs (4)–(6) are, respectively

$$J^{\min}(\mathcal{S},1) = 2\left(n+1+\sqrt{n^2+2n-3}\right)^{-1} \tag{48}$$
$$J^{\max}(\mathcal{S},1) = 2\left(n+1-\sqrt{n^2+2n-3}\right)^{-1} \tag{49}$$

and

$$J^{\text{avg}}(\mathcal{S},1) = 2 - \frac{1}{n}. \tag{50}$$

Proof: The Dirichlet matrix for an influencing agent attached to the central node of $\mathcal{S}$, v_1, is

$$-A(\mathcal{S},1) = \begin{bmatrix} n & -\mathbf{1}^T \\ -\mathbf{1} & I \end{bmatrix}. \tag{51}$$

Examining the identity $-A(\mathcal{S},1)v = \lambda v$, we note that there are $n-2$ eigenvectors of the form $v = [0\ \alpha]^T$, where $\alpha \in \mathbb{R}^{n-1}$ and $\sum \alpha_i = 0$ corresponding to $\lambda = 1$, and two eigenvectors of the form $v = [1-\lambda\ \ \mathbf{1}^T]^T$, where

$$\lambda = \frac{1}{2}\left(1+n \pm \sqrt{n^2+2n-3}\right). \tag{52}$$

■

Proposition A.4: For the n-node star graph $\mathcal{S}$, with influencing node attached to the central node $v_1 \in \mathcal{S}$, the controllability gramian of (2) is

$$P(\mathcal{S},1) = \frac{1}{2(n+1)}\left(\mathbf{1}\mathbf{1}^T + e_1 e_1^T\right) \tag{53}$$

with $\lambda_j(P(\mathcal{S},1)) = 0$, for $j=1,\ldots,n-2$ and

$$\lambda_j\left(P(\mathcal{S},1)\right) = \frac{1}{4}\left(1 \pm \frac{\sqrt{n^2-2n+5}}{n+1}\right) \tag{54}$$

for $j = n-1, n$, respectively.

Proof: The proof follows from verifying that (53) satisfies the Lyapunov equation with $B = e_1$. The gramian and its corresponding eigenvalues are identical to those of the complete graph with an attached node. ■

Proposition A.5: For the n-node complete graph $\mathcal{K}$, with an influencing node attached to any node $v_i \in \mathcal{K}$, the minimum, maximum, and average performance costs (4)–(6) are, respectively

$$J^{\min}(\mathcal{K},i) = 2\left(n+1+\sqrt{n^2+2n-3}\right)^{-1} \tag{55}$$
$$J^{\max}(\mathcal{K},i) = 2\left(n+1-\sqrt{n^2+2n-3}\right)^{-1} \tag{56}$$

and

$$J^{\text{avg}}(\mathcal{K},i) = \left(1+\frac{2}{n}-\frac{2}{n^2}\right). \tag{57}$$

Proof: The Dirichlet matrix for an influencing agent attached to v_1, an arbitrary node of $\mathcal{K}$, is

$$-A(\mathcal{K},1) = \begin{bmatrix} n & -\mathbf{1}^T \\ -\mathbf{1} & -\mathbf{1}\mathbf{1}^T + nI \end{bmatrix}. \tag{58}$$

Examining the identity $-A(\mathcal{K},1)v = \lambda v$, there are $n-2$ eigenvectors of the form $v = [0\ \alpha]^T$, where $\alpha \in \mathbb{R}^{n-1}$ and $\sum \alpha_i = 0$ corresponding to $\lambda = n$ and two eigenvalues of the form $v = [1-\lambda\ \ \mathbf{1}^T]^T$, where

$$\lambda = \frac{1}{2}\left(1+n \pm \sqrt{n^2+2n-3}\right). \tag{59}$$

Due to the symmetry of $\mathcal{K}$, the cost incurred by the attached node to steer the network is independent of where it attaches. ■

Proposition A.6: For the n-node complete graph $\mathcal{K}$, with influencing node attached to any node $v_1 \in \mathcal{K}$, the controllability gramian of (2) is the same as that of the n-node star graph $\mathcal{S}$, with influencing node attached to the central node $v_1 \in \mathcal{S}$ (53). ■

Proof: The proof follows directly from Proposition 3.3. ■

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Fourier-Hermite Kalman Filter

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***Abstract*—In this note, we shall present a new class of Gaussian filters called Fourier-Hermite Kalman filters. Fourier-Hermite Kalman filters are based on expansion of nonlinear functions with the Fourier-Hermite series in same way as the traditional extended Kalman filter is based on the Taylor series. The first order truncation of the Fourier-Hermite series gives the previously known statistically linearized filter.**

***Index Terms*—Extended Kalman filtering (EKF), Fourier-Hermite series, nonlinear KF, statistical linearization.**

I. INTRODUCTION

The *Kalman filter* (KF) [1] is concerned with estimation of the dynamic state from noisy measurements in the class of estimation problems where the dynamic and measurement processes can be approximated by linear Gaussian state space models. The KF is also applicable to linear state space models with a wide range of non-Gaussian noise distributions [2]. General filtering theory for nonlinear and non-Gaussian models was already presented in [3], [4], but in practice, numerical solutions derived as approximations to the general theory are usually computationally more demanding than the Gaussian approximations derived as extensions to the KF. The Taylor series based *extended Kalman filter* (EKF) [4], and the Gaussian describing function based *statistically linearized filter* (SLF) [5] are the classical Gaussian approximation based extensions of the KF to nonlinear dynamic and measurement models.

Recently, numerical integration based sigma point filters [6]–[10], have been introduced as alternatives to the classical linearization based methods. Many of the sigma point methods can also be interpreted as numerical approximations to the SLF [11]–[13]. In this note, we shall take the opposite approach from the sigma point methods—instead of approximating the SLF we shall develop higher order approximations by extending SLF. In numerical comparison, the new approach is found to give similar results as the sigma point methods. The advantage of the new method compared to the sigma point methods is that it provides a closed form approximation instead of applying a numerical method directly. The implementation of the closed form solution can be more efficient. If closed form solution is not possible for some part of the problem then it is still possible to use the numerical sigma point approach for that part.

In this technical note, we shall introduce a new class of filters that we call Fourier-Hermite Kalman filters (FHKF). The filters are based on a finite truncation of the Fourier-Hermite series in a similar way as the EKF is based on a truncation of the Taylor series. The first order truncation gives the previously known SLF [5] in a similar way as the first order truncation of the Taylor series gives the basic EKF. The new approach also makes it possible to use higher order truncation of the Fourier-Hermite series similar to the second order EKF. Due to the orthogonality of the Hermite polynomials, any order truncation is almost as easy to use as the first order truncation and gives the best possible

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