Combinatorially Thinking

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Philosophy. We want to *construct* our mathematical understanding. To this end, our goal is to situate our problems in concrete counting contexts. Most mathematicians appreciate clever combinatorial proofs. But faced with an identity, how can *you* create one?

This course will provide you with some useful combinatorial interpretations, lots of examples, and the challenge of finding your own combinatorial proofs. Throughout the next two weeks, your mantra should be to *keep it simple.*

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1 Basic Tools

1.1 Some Combinatorial Interpretations

First, we need combinatorial interpretations for the objects occurring in our identities. While there are many possible interpretations, only one is presented for each mathematical object—trying, of course, to *keep it simple.*

I have included at least one method to compute each object for completeness though we will rarely rely on computation.

$n!$— *factorial*

Combinatorial: The ways to arrange numbers $1, 2, 3, \ldots, n$ in a line.

Computational: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

$\binom{n}{k}$—*binomial coefficient, n choose k*

Combinatorial: The ways to select a subset containing k elements from the set $[n] = \{1, 2, 3, \ldots, n\}$.

Computational: $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\left(\binom{n}{k}\right)$—*n multichoose k*

Combinatorial: The ways to cast k votes for elements from the set $[n] = \{1, 2, 3, \ldots, n\}$.

Computational: $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$

$\left[{n \atop k}\right]$—*(unsigned) Stirling number of the first kind*

Combinatorial: The ways to arrange n people around k identical (nonempty) circular tables.

Computational: Recursively $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ and $n \geq 1$ $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$. For $k \geq 2$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n+1) \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

$\left\{ {n \atop k} \right\}$—*Stirling number of the second kind*

Combinatorial: The ways to distribute n people into k identical (nonempty) rooms.

Computational: Recursively $\left\{ {n \atop 0} \right\} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ and $n \geq 1$ $\left\{ {n \atop 1} \right\} = 1$. For $k \geq 2$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

f_n—*the nth Fibonacci number*

Combinatorial: The ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes.

Computational: $f_0 = 1, f_1 = 1$, and for $n \geq 2$ $f_n = f_{n-1} + f_{n-2}$.

$$\text{or } f_n = \frac{1}{\sqrt{5}} \left[\phi^{n+1} - \left(\frac{-1}{\phi} \right)^{n+1} \right]$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

WARNING: You might be used to the Fibonacci numbers defined in the more traditional way $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ $F_n = F_{n-1} + F_{n-2}$.

L_n—*the nth Lucas number*

Combinatorial: The ways to tile a circular $1 \times n$ board using 1×1 "squares" and 1×2 "dominoes".

Computational: $L_0 = 2, L_1 = 1$, and for $n \geq 2$ $L_n = L_{n-1} + L_{n-2}$.

$$\text{or } L_n = \phi^n + \left(\frac{-1}{\phi} \right)^n.$$

G_n—*the nth Gibonacci number*

Combinatorial: The ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes where the first tile is distinguished. There are G_1 choices for a leading square and G_0 choices for a leading domino.

Computational: G_0 and G_1 are given and for $n \geq 2$ $G_n = G_{n-1} + G_{n-2}$.

$$\text{or } G_n = \alpha \phi^n + \beta \left(\frac{-1}{\phi} \right)^n$$

where $\alpha = (G_1 + G_0/\phi)/\sqrt{5}$ and $\beta = (\phi G_0 - G_1)/\sqrt{5}$.

D_n—*the nth Derangement number*

Combinatorial: The ways to arrange $1, 2, \ldots, n$ in a line so that no number lies in its natural position.

Computational: $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$.

C_n—*the nth Catalan number*

Combinatorial: The number of lattice paths from $(0, 0)$ to (n, n) using "right" and "up" edges and staying below the line $y = x$.

Computational: $C_n = \frac{1}{n+1}\binom{2n}{n}$

$[a_0, a_1, \ldots, a_n]$—*the finite continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}$$

Combinatorial:

Numerator: The ways to tile a $1 \times n+1$ board using squares and dominoes where cell i can contain half a domino or as many as a_i squares, $0 \leq i \leq n$.

Denominator: The ways to tile a $1 \times n$ board using squares and dominoes where cell i can contain half a domino or as many as a_i squares, $1 \leq i \leq n$.

Computational: Attack with algebra to rationalize the complex fraction.

$\det(A)$—*the determinant of the $n \times n$ matrix $A = \{a_{ij}\}$.*

Combinatorial: The signed sum of nonintersecting n-routes in a directed graph with n origins, n destinations, and a_{ij} directed paths from origin i to destination j.

Computational: $\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

Example: $\det \begin{bmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{bmatrix} = 1{\cdot}8{\cdot}2 + 2{\cdot}21{\cdot}0 + 5{\cdot}5{\cdot}1 - 0{\cdot}8{\cdot}5 - 5{\cdot}2{\cdot}2 - 1{\cdot}1{\cdot}21 = 0.$

1.2 Counting Technique 1: Ask a Question and Answer Two Ways

We will use one of two techniques to *count* an identity. The first poses a question and then answers it in two different ways. One answer is the left side of the identity; the other answer is the right side. Since both answers solve the same counting question, they must be equal.

Identity 1 *For* $n \geq 1$,

$$\sum_{k=1}^{n-1} k \cdot k! = n! - 1.$$

Question: The number of ways to arrange $1, 2, 3, \ldots, n$ except for

Answer 1:

Answer 2:

Identity 2 *For* $k, n \geq 0$,

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Question: How many ways can we allocate k votes to n candidates?

Answer 1:

Answer 2:

1.3 Counting Technique 2: Description-Involution-Exception (DIE)

The second technique is to create two sets, count their sizes, and find a correspondence between them. The correspondence could be one-to-one, many-to-one, almost one-to-one, or almost many-to-one.

Identity 3 *For $k, n \geq 0$,*

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Description:

Set 1:

Set 2:

Involution:

Exception:

Identity 4 *For $n \geq 0$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Description:

Set 1:

Set 2:

Involution:

Exception:

What happens is we change the upper index of the summation to something smaller than n? larger than n?

2 Binomial Identities

2.1 Working Together

Let's use these techniques to prove some identities.

Identity 5 *For* $0 \le k \le n$

$$n! = \binom{n}{k} k!(n-k)!$$

Identity 6 The Binomial Theorem. *For* $n \ge 0$,

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n.$$

Identity 7 *For* $n \ge 0$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

2.2 On Your Own

Identity 8 *For* $0 \leq k \leq n$, *(except* $n = k = 0$*),*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The technique above can be modified to prove:

Identity 9 *For* $n \geq 0, k \geq 0$, *(except* $n = k = 0$*),* $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right).$

Identity 10 *For* $n \geq k \geq 1$, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}.$

Identity 11 *For* $n \geq k \geq 1$, $\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}.$

Identity 12 *For* $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Identity 13 *For* $n \geq 1$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Identity 14 *For* $n \geq q \geq 0$,

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

Identity 15 *For* $n \geq m, k \geq 0$,

$$\sum_{j} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}.$$

Identity 16 *For* $n \geq k \geq j \geq 0$,

$$\sum_{m} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$$

Identity 17 *For nonnegative integers* $k_1, k_2, \ldots, k_n$, *let* $N = \sum_{i=1}^{n} \binom{k_i}{2}$. *Then*

$$\sum_{1 \leq i < j \leq n} \binom{k_i}{2}\binom{k_j}{2} + 3\sum_{i=1} n\binom{k_i+1}{4} = \binom{N}{2}.$$

2.3 What's the Parity of $\binom{n}{k}$?

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

Serpinski-like Triangle

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 0 1 1
1 0 1 0 0 0 0 0 1 0 1
1 1 1 1 0 0 0 0 1 1 1 1

Theorem. *For $n \geq 0$, the number of odd integers in the nth row of Pascal's triangle is equal to 2^b where $b =$the number of 1s in the binary expansion of n.*

Question. How many odd numbers occur in the 76^{th} row of Pascal's triangle?

Lemma. *The parity of $\binom{n}{k}$ will be the same as the parity of the number of palindromic sequences with k ones and $n-k$ zeros.*

Description. Binary sequences with k ones and $n-k$ zeros.

Involution.

Exception.

Ans: $76 = (1001100)_2$. *Eight odd numbers.*

Count palindromic sequences with k ones and $n-k$ zeros.

$\binom{even}{odd}$

$\binom{even}{even}$

$\binom{odd}{even}$

$\binom{odd}{odd}$

Consequence. If n is even and k is odd, then $\binom{n}{k}$ is even, otherwise, $\binom{n}{k}$ has the same parity as $\binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}$ where we round $n/2$ and $k/2$ down to the nearest integer, if necessary.

Examples. Compute the parity:

$$\binom{76}{15} \qquad \binom{76}{36} \qquad \binom{76}{12}$$

Think binary!

$$\binom{(1001100)_2}{(0001111)_2} \qquad \binom{(1001100)_2}{(0100100)_2} \qquad \binom{(1001100)_2}{(0001100)_2}$$

Only way to have an odd number is if the 1s in the binary representation of k are directly below 1s in binary representation of n.

It tells us *exactly* which numbers produce odd binomial coefficients:

binary representation	k
1001100	76
1001000	72
1000100	68
1000000	64
0001100	12
0001000	8
0000100	4
0000000	0

Extension. There a similar procedure to determine the remainder of $\binom{n}{k}$ when divided by *any* prime p?

Lucas' Theorem. *For any prime p, we can determine the remainder of $\binom{n}{k}$ when divided by p from the base p expansions of n and k. If*

$$\begin{aligned} n &= b_t p^t + b_{t-1} b^{t-1} + \cdots + b_1 p^1 + b_0 \\ k &= c_t p^t + c_{t-1} b^{t-1} + \cdots + c_1 p^1 + c_0 \end{aligned}$$

then $\binom{n}{k}$ and $\binom{b_t}{c_t}\binom{b_{t-1}}{c_{t-1}}\cdots\binom{b_1}{c_1}\binom{b_0}{c_0}$ have the same remainder when divided by p.

Example. Calculate the remainder of $\binom{97}{35}$ when divided by 5.

Ans: $\binom{97}{35} \equiv \binom{3}{1}\binom{4}{2}\binom{2}{0} \equiv 3 \bmod 5$