3 Fibonacci Identities

3.1 Tiling with Squares and Dominoes

Definition. Let f_n count the ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes.

Questions to explore.

1. Compute f_4, f_5, and f_6.

2. Of the square-domino tilings of the 1×6 board, how many are *breakable* after the second cell? How many have a single domino covering cells two and three?

3. For a board of length n (henceforth called an n-board),

   (a) how many start with a square?

   (b) how many start with a domino?

   (c) how many use exactly k dominoes (and what values make sense for k?)

4. What should f_0 be? What should f_{-1} be?

3.2 Working Together

Identity 18 *For* $n \geq 0$,

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_n.$$

Identity 19 *For* $n \geq 0$,

$$f_{n-1}^2 + f_n^2 = f_{2n}$$

Identity 20 *For* $n \geq 0$,

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}.$$

Identity 21 *For* $n \geq 2$,

$$3f_n = f_{n+2} + f_{n-2}.$$

3.3 On Your Own

Identity 22 *For* $n \geq 0$

$$f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

Identity 23 *For* $n \geq 0$,

$$f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}.$$

Identity 24 *For* $n \geq 0$,

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^n$$

Identity 25 *For* $n \geq 2$,

$$2f_n = f_{n+1} + f_{n-2}.$$

Identity 26 *For* $n \geq 0$,

$$\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-i}{j}\binom{n-j}{i} = f_{2n+1}.$$

Identity 27 *For* $n \geq 0$,

$$\sum_{k=0}^{n} kf_{n-k} = f_{n+3} - (n+3)$$

Identity 28 *For* $n \geq 0$,

$$f_{2n-1} = \sum_{k\geq 0}\binom{n}{k}f_{k-1}.$$

Identity 29 *For* $n \geq 0$,

$$2^n f_{2n-1} = \sum_{k\geq 0}\binom{n}{k}f_{3k-1}.$$

Identity 30 *If* $m|n$, *then* $f_{m-1}|f_{n-1}$ *(i.e.* $F_m|F_n$*)*

> *Hint: If* $n = qm$, *count* $(n-1)$*-tilings by considering the smallest value* j *such that the tiling is breakable at cell* $jm-1$. *(Why must such a cell exist?)*

3.4 Combinatorial Proof of Binet's Formula

Is it possible to construct a combinatorial proof of identities involving irrational quantities? For example, using the standard Fibonacci Society definition $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, we have Binet's classic formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

A combinatorial proof is possible if we introduce *probability.* Let $\phi = \frac{1+\sqrt{5}}{2}$.

FACTS:

- $\left(\frac{1-\sqrt{5}}{2}\right) = -1/\phi$.
- $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$.
- Equivalent identity (since $F_n = f_{n-1}$).

$$f_n = \frac{1}{\sqrt{5}}\left[\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right].$$

Tile an infinite board by independently placing squares and dominoes, one after the other. At each decision, place a square with probability $1/\phi$ or a domino with probability $1/\phi^2$. The probability that a tiling begins with any particular length n sequence is $1/\phi^n$. Let q_n be the probability that an infinite tiling is breakable at cell n.

$$q_n = \tag{1}$$

Question. What is the probability that an infinite tiling is breakable at cell n?

Answer 1. q_n

Answer 2. One minus the probability that it is not breakable at cell n.

Remember $q_0 = 1$.

Unravel the recurrence in (1) to get

$$q_n = 1 - \frac{1}{\phi^2} + \frac{1}{\phi^4} - \frac{1}{\phi^6} + \cdots + \left(\frac{-1}{\phi^2}\right)^n.$$

Sum the geometric series.

Thus $f_n = \phi^n q_n =$