4 Generalizations—Lucas, Gibonacci, and Linear Recurrences

4.1 Motivating Generalization

Revisit Identity 28. *For $n \geq 0$,*

$$f_{2n-1} = \sum_{k \geq 0} \binom{n}{k} f_{k-1}.$$

Question: How many $(2n-1)$-tilings?

Answer 1: f_{2n-1}

Answer 2: Let k be the number of squares among the first n tiles.

Was the -1 really necessary in f_{2n-1} or f_{k-1} to make this argument fly?

Identity 31 *For $n \geq 0$, $p \geq -1$,*

$$f_{2n+p} = \sum_{k \geq 0} \binom{n}{k} f_{k+p}.$$

Identity seems independent of "initial conditions". Why not consider a general Fibonacci sequence (a.k.a. Gibonacci sequence)?

Definition. Let G_0 and G_1 be specified and for $n \geq 2$ define $G_n = G_{n-1} + G_{n-2}$.

The first few terms in a Gibonacci sequence are
G_0, G_1,

4.2 Weighting the Tilings

To transform the square-domino tilings of an n-board that give us the Fibonacci number f_n, to a Gibonacci number G_n we weight the tiling as follows:

weight assigned to tiling ending in a *square*:	G_1
weight assigned to tiling ending in a *domino*:	G_0

Let w_n equal the sum of the weights of the length n-tilings.
Questions to explore.

1. Compute w_1, w_2, and w_3.

2. For a board of length n (henceforth called an n-board),

   (a) what is the total weight attributable to tilings that start with a square?

   (b) what is the total weight attributable to tilings that start with a domino?

3. What should w_0 be?

So $w_n = G_n$ **for** $n \geq 0$.

Combinatorial Interpretation. For a Gibonacci sequence $G_0, G_1, G_2, \ldots$, the Gibonacci number G_n is the total weight of all square-domino tilings of length n where tilings ending withe a square have weight G_1 and tilings ending with a domino have weight G_0.

4.3 Working Together

Identity 32 *For* $m, n \geq 0$,

$$G_{m+n} = G_m f_n + G_{m-1} f_{n-1}.$$

Identity 33 *For* $n \geq 0$,

$$G_0 + G_1 + G_2 + \cdots + G_n = G_{n+2} - G_1.$$

Identity 34 *For* $n \geq 1$,

$$G_{n+1} G_{n-1} - G_n^2 = (-1)^n (G_1^2 - G_0 G_2).$$

Identity 35 *For* $n \geq 0$,

$$G_{2n} = \sum_{k \geq 0} \binom{n}{k} G_k.$$

Identity 36 *For* $n \geq 2$,

$$G_{n+2} + G_{n-2} = 3G_n.$$

4.4 On Your Own

Of special note is the **Lucas sequence**, the so-called companion sequence to the Fibonaccis. Here $L_0 = 2$, $L_1 = 1$ and for $n \geq 2$ $L_n = L_{n-1} + L_{n-2}$. While weighted tilings work fine, we could reinterpret Lucas numbers as circular tilings. (See Section 1.1.)

Identity 37 *For $n \geq 1$,*

$$G_n = G_0 f_{n-2} + G_1 f_{n-1}.$$

Identity 38 *For $n \geq 0$,*

$$G_1 + \sum_{k=1}^{n} G_{2k} = G_{2n+1}.$$

Identity 39 *For $n \geq 0$,*

$$G_0 G_1 + \sum_{k=1}^{n} G_k^2 = G_n G_{n+1}.$$

Identity 40 *For $n \geq 0$,*

$$5 f_n = L_n + L_{n+2}$$

Identity 41 *For $n \geq 0$,*

$$L_n^2 = L_{2n} + (-1)^n \cdot 2.$$

Identity 42 *Let $G_0, G_1, G_2, \ldots$ and $H_0, H_1, H_2, \ldots$ be Gibonacci sequences. Then for $0 \leq m \leq n$,*

$$G_m H_n - G_n H_m = (-1)^m (G_0 H_{n-m} - G_{n-m} H_0).$$

Identity 43 *For $n \geq 0$,*

$$2^n G_{2n} = \sum_{k \geq 0} \binom{n}{k} G_{3k}.$$

4.5 Weighting tiles individually

For the Gibonacci numbers, we weighted the tiling based on the last tile used (G_0 if it ended in a domino and G_1 if it ended in a square.) We have seen, that sometimes it was simpler to think of the weight as being attached to the last tile itself. We could do this if the rest of the board had "weight 1" and the weight obtained by concatenating two boards is the **product** of the weights of its parts. This leads to the following idea:

Definition. The *weight* of a tiling T, denoted $w(T)$, is the product of the weight of the individual tiles. The *total weight* for tilings of length n, denoted t_n, is the sum of the weights of all tilings of length n.

Example. Suppose we tiling n-boards with squares of weight s and dominoes of weight d. Find t_1, t_2, t_3, t_4.

Can you find a recurrence to express t_n?

What are the initial conditions for the situation described above?

For general linear recurrences of order 2

Combinatorial Interpretation. Let s, d, a_0, a_1 be given and for $n \geq 2$, define

$$a_n = sa_{n-1} + da_{n-2}.$$

For $n \geq 1$, a_n is the total weight of length n-tilings created from weight s squares and weight d dominoes except for the last tile, which may be a weight a_1 square or a weight da_0 domino.

For the three identities below, we will assume ideal initial conditions: suppose $a_0 = 1, a_1 = s$ and for $n \geq 2$, $a_n = sa_{n-1} + ta_{n-2}$.

Identity 44 *For $m, n \geq 1$,*

$$a_{m+n} = a_m a_n + t a_{m-1} a_{n-1}.$$

Identity 45 *For $n \geq 2$,*

$$a_n - 1 = (s-1)a_{n-1} + (s+t-1)\sum_{k=1}^{n-1} a_{k-1}.$$

Identity 46 *For any $1 \leq c \leq s$, for $n \geq 0$,*

$$a_n - c^n = (s-c)a_{n-1} + ((s-c)c+t)\sum_{k=1}^{n-1} a_{k-1}c^{n-1-k}.$$

4.6 A Gibonacci Magic Trick

Secretly write a positive integer in row 1 and another positive integer in row 2.

row	integers
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Next add those numbers together and put the sum in row 3. Add row 2 and row 3 and place the answer in row 4. Continue in this fashion until numbers are in rows 1 through 10. Now using a calculator, if you wish, add all the numbers in rows 1 through 10 together.

While the volunteer is adding, the mathemagician glances at the sheet of numbers for just a second and instantly reveals the sum. How?

Suppose $G_0 = x$ and $G_1 = y$ are our secret numbers....

row	integers
1	x
2	y
3	
4	
5	
6	
7	
8	
9	
10	

As a final flourish to the mathemagician's performance he adds, "Now using a calculator, divide the number in row 10 by the number in row 9 and announce the first three digits of your answer. What's that you say? 1.61? Now turn over the paper and look what I have written." The back of the paper says, "I predict the number 1.61."

Why does the ratio work?

For any two fractions $\frac{a}{b} < \frac{c}{d}$ with positive numerators and denominators, it is easy to show that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

What are the implications of this for (row 10)/(row 9)?