5 Continued Fractions

5.1 Introductions and Explorations

Definition. Given integers $a_0 \geq 0$, $a_1, \leq 1, a_2 \geq 1, \ldots, a_n \geq 1$, define *finite continued fraction*, denoted $[a_0, a_1, \ldots, a_n]$, to be the fraction in lowest terms for

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Compute: $[3, 4, 5, 2]$ and $[2, 5, 4, 3]$

Find finite continued fractions that represent 37/17 and 9/5.

Understanding Continued Fractions Through Algebra

Suppose $[a_0, a_1, \ldots, a_n] = \dfrac{p(a_0, a_1, \ldots, a_n)}{q(a_0, a_1, \ldots, a_n)}$. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = a_0 + \frac{1}{\frac{p(a_1, \ldots, a_n)}{q(a_1, \ldots, a_n)}}$$

Initial conditions? $p(a_0)$ $q(a_0)$ $p(a_0, a_1)$ $q(a_0, a_1)$

This yields the following relations for the numerator and denominator of finite continued fractions:

$$\begin{aligned} p(a_0, a_1, \ldots, a_n) &= a_0 p(a_1, \ldots, a_n) + p(a_2, \ldots, a_n) \\ q(a_0, a_1, \ldots, a_n) &= p(a_1, \ldots, a_n) \end{aligned}$$

Counting Context

Let $K(a_i, a_{i+1}, \ldots, a_j)$ be the total weight of the square-domino tilings of an $1\times(j-i+1)$-board (with cell indexed from i to j) where dominoes have weight 1 and a square on cell k has weight a_k, $i \le k \le j$.

Compute $K(3, 4, 5, 2)$.

Find a recurrence for $K(a_0, a_1, a_2, \ldots, a_n)$ based on the first tile.

Initial conditions:

$K(a_0) =$

$K(a_0, a_1) =$

Combinatorial Consequence. The finite partial fraction

$$[a_0, a_1, \ldots, a_n] = \frac{K(a_0, a_1, \ldots, a_n)}{K(a_1, \ldots, a_n)}.$$

5.2 Working Together

Theorem 2 *Reversing the entries in a finite continued fraction does not change the value of the numerator.*

Definition. An *infinite continued fraction* $[a_0, a_1, a_2, \ldots] = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n]$. The finite approximation $[a_0, a_1, \ldots, a_n] = r_n = \frac{p_n}{q_n}$ is called a *convergent* of the continued fraction.

Identity 47 *The difference between consecutive convergents of* $[a_0, a_1, \ldots]$ *is:*

$$r_n - r_{n-1} = \frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

Equivalently, after multiplying both sides by $q_n q_{n-1}$*, we have*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

One consequence is that p_n/q_n *is in lowest terms since we have an integer combination of* p_n *and* q_n *that equals* ± 1*.*

Identity 48 *The difference between every other convergent of* $[a_0, a_1, \ldots]$ *is:*

$$r_n - r_{n-2} = \frac{(-1)^n a_n}{q_n q_{n-2}}.$$

Equivalently, after multiplying both sides by $q_n q_{n-2}$*, we have*

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

How can you use the last two identities to see that infinite continued fractions always converge (and to an irrational number at that!)

Identity 49 $[a_0, a_1, \ldots, a_{n-1}, 1] = [a_0, a_1, \ldots, a_{n-1} + 1]$

5.3 On Your Own

Prove the following by direct combinatorial argument.

Identity 50 *For* $n \geq 0$, $[a_0, a_1, \ldots, a_n] = [1, 1, \ldots, 1] = \frac{f_{n+1}}{f_n}$

Identity 51 *For* $n \geq 0$, $[a_0, a_1, a_2, \ldots, a_n] = [3, 1, 1, \ldots, 1] = L_{n+2}/f_n$.

Identity 52 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [1, 1, \ldots, 1, 3] = L_{n+2}/L_{n+1}$.

Identity 53 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [4, 4, \ldots, 4, 3] = f_{3n+3}/f_{3n}$.

Identity 54 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [4, 4, \ldots, 4, 4] = f_{3n+5}/f_{3n+2}$.

Problem. Use Identities 47 and 48 to show that infinite continued fractions are well defined (i.e. the defining limit actually exists).

Problem. If an infinite continued fraction $[a_0, a_1, \ldots]$ converges to r, can you show that r must be irrational?

> *Hint: Show that* $0 < |r - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$. *Then assume that* r *is rational to arrive at a contradiction.*

5.4 Primes of form $4m + 1$

Theorem 3 *Any prime of the form $4m + 1$ can be (uniquely) written as the sum of the squares of two positive integers.*

Examples. Write 5, 13, and 17 as the sum of two squares of two positive integers.

Proof. Suppose that $p = 4m + 1$ is prime and consider the continued fraction expansions of

$$\frac{p}{1}, \frac{p}{2}, \dots, \frac{p}{2m}.$$