# 6 Alternating Sums

## 6.1 Working Together

Alternating sums (or sums where the sign of the summand alternates between positive and negative), are almost exclusively counted using the D.I.E. Method. The goal is to describe two sets—one that is created from the positive "stuff" and the second from the negative "stuff", combinatorially interpret the quantity that changes sign, pair of positive and negative "stuff", and count the exceptions. The first example of this technique was proving Identity 4 and its generalization

**Identity 55 Generalization of Identity 4.** *For $m \geq 0$ and $n > 0$*

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}.$$

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**Identity 56** *For $n \geq 0$,*

$$\sum_{k=0}^{n}(-1)^k f_k = 1 + (-1)^n f_{n-1}.$$

*Note $f_{-1} = 0$.*

**Interpret Quantity.** $f_k$

**Set $P$.**

**Set $N$.**

**Correspondence.**

**Exceptions.**

*Can you figure out the Gibonacci version of this identity?*

**Identity 57** *For $n \geq 0$,*

$$\sum_{k=0}^{n} (-1)^k k \binom{n}{k} = 0.$$

**Interpret Quantity.** $k\binom{n}{k}$

**Set $P$.**

**Set $N$.**

**Correspondence.**

**Exceptions.**

**Identity 58**

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

**Interpret Quantity.** $\frac{n!}{k!}$

**Set $P$.**

**Set $N$.**

**Correspondence.**

**Exceptions.**

## 6.2 On Your Own

**Identity 59** *For* $0 \le m < n$,

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{m} (-1)^k = 0.$$

**Identity 60** *For* $n \ge 0$, $\sum_{k\ge 0} (-1)^k \binom{n-k}{k} = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 (\text{mod } 6) \\ 0 & \text{if } n \equiv 2 \text{ or } 5 (\text{mod } 6) \\ -1 & \text{if } n \equiv 3 \text{ or } 4 (\text{mod } 6) \end{cases}$

**Identity 61** *For* $n \ge 0$, $\sum_{k\ge 0} (-1)^k \binom{n-k}{k} 2^{n-2k} = n + 1.$

**Identity 62**

$$\sum_{k\ge 0} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \sum_{j\ge 0} x^{n-j} y^j$$

**Identity 63** *For* $n \ge 0$, $\sum_{k=0}^{n} (-1)^k f_{2k} = (-1)^n f_n^2.$

**Identity 64** *For* $n \ge 3$,

$$\sum_{i=0}^{n} (-1)^i i f_i = (-1)^n (n f_{n-1} + f_{n-3}) + 1$$

**Identity 65** *For* $0 \le k \le n$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k = n! \cdot \delta_{k,n}$$

## 6.3 Extend and Generalize by Playing with Parameters

**Identity 66**

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1} = 0.$$

**Interpret Quantity.** $\binom{n}{r}\binom{2n-2r}{n-1}$

**Set P.**

**Set N.**

**Correspondence.**

**Exceptions.**

**Questions to consider for generalizations**

- What role does $n-1$ play in $\binom{2n-2r}{n-1}$?
- What can you say if you replace $n-1$ with an arbitrary $m < n$?
- What can you say if you replace $n-1$ with an arbitrary $m > n$?
- Rather than considering $n$ pairs, what is we painted $n$ triples? quadruples? $k$-tuples?
- Does the summation have to go all the way from $r = 0$ to $r = n$? Can we compute a partial sum...say to some fixed $s < n$?

**Identity 67** *For* $0 \le m < n$,

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{2n-2r}{m} = 0.$$

**Identity 68** *For* $n, m \ge 0$,

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{2n-2r}{m} = 2^{2n-m} \binom{n}{m-n}.$$

**Identity 69** *For* $0 \le m < n$ *and* $k \ge 1$,

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn-kr}{m} = 0.$$

**Identity 70**

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn-kr}{n} = k^n.$$

**Identity 71**

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn-kr}{n+1} = nk^{n-1} \binom{k}{2}.$$

**Identity 72** *For* $0 \le m < s \le n$,

$$\sum_{r=0}^{s} (-1)^r \binom{s}{r} \binom{2n-2r}{m} = 0.$$

**Identity 73** *For* $0 \le s \le n$,

$$\sum_{r=0}^{s} (-1)^r \binom{s}{r} \binom{2n-2r}{s} = 2^s.$$