6.4 Beyond DIE—a.k.a. "What happens after you DIE?"

This identity requires the twist of iterated involutions.

Identity 74

$$\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{n-1}{j-1}\binom{n-1}{k-1}\binom{j+k}{j} = 2.$$

Interpret Quantity. $\binom{n-1}{j-1}\binom{n-1}{k-1}\binom{j+k}{j}$

Set P.

Set N.

Correspondence.

In the preceding analysis, elements 1 and $n+1$ represented the guaranteed members of X and Y respectively. There is no reason to believe we are restricted to specifying only one member of each set. Why not two? three? Why not specify a members of X and b members of Y? Originally, X and Y were selected from disjoint sets of size n. Did they have to be the same size? Why not choose X from an n-set and Y from an m-set? With very little additional effort, you can modify the above description, involutions, and exceptions to obtain the following generalization:

Identity 75

$$\sum_{j=a}^{n}\sum_{k=b}^{m}(-1)^{j+k}\binom{n-a}{j-a}\binom{m-b}{k-b}\binom{j+k}{j} = \binom{a+b}{n-m+b}.$$

6.5 Binet Revisited

You are now prepared for one last attack of Binet's formula through finite weighted colored tilings. If $\phi = \frac{1+\sqrt{5}}{2}$, then $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ and Binet's formula can be expressed as

$$f_n = \frac{1}{\sqrt{5}}(\phi^{n+1} - \overline{\phi}^{n+1}).$$

Definition. Let B_n be the total weight of a square domino tiling of a $1 \times n$ board where weights of tiles are assigned as follows:

tile type	*tile location*	*weight assigned*
domino	anywhere	1
white square	any cell ≥ 2	ϕ
white square	cell 1	$\frac{\phi^2}{\sqrt{5}}$
black square	any cell ≥ 2	$\overline{\phi}$
black square	cell 1	$\frac{-\overline{\phi}^2}{\sqrt{5}}$

Compute B_0, B_1, B_2

For $n \geq 2$, find a recurrence for B_n based on the weight of the last tile.

So $B_n = f_n$.

"Involution" Gather tilings that add to zero:

Exceptions

7 Determinants

Calculate the following determinants:

$$\begin{vmatrix} 1 & 1 \\ 10 & 11 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{vmatrix} =$$

In general

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

What about

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

7.1 The Big Formula

The determinant of an $n \times n$ matrix $A = \{a_{ij}\}$ is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

A permutation σ of S_n is an arrangement of $[n]$ (there are $n!$ of these). In terms of a matrix, it corresponds to an $n \times n$ matrix of 1s and 0s where there is exactly one 1 in each row and column.

Example. Find the 6×6 matrices corresponding to the following permutations.

123456 145236 534621

The $\operatorname{sign}(\sigma)$ is ± 1 depending on *parity* of the number of row exchanges (transpositions) needed to transform it to the identity. (Even $\to +1$; Odd $\to -1$.)

Permutations of S_3 For each permutation matrix below determine the corresponding permutation of [3] and the sign of the permutation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

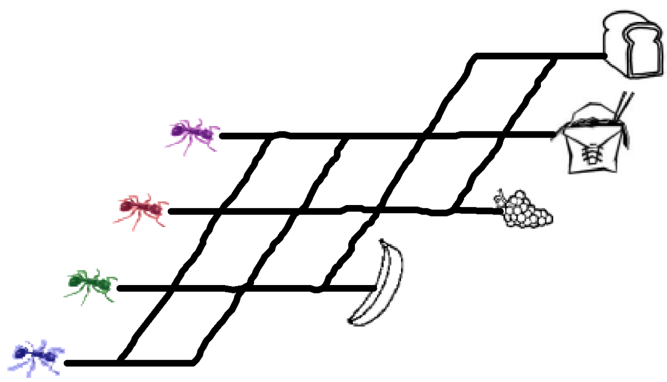
Check your Understanding. Use the big formula to compute

$$\det(A) = \begin{vmatrix} 2 & 5 & 14 & 23 \\ 1 & 3 & 9 & 15 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{vmatrix} =$$

This is a nice answer. Our goal will be to "see" why.

7.2 Matrices from Determined Ants

Given a directed graph with n origins (the ants) and n destinations (the food)



create a matrix A where entry a_{ij} represents the number of paths for ant i to get to food j.

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} & & 14 & 23 \\ & & 9 & 15 \\ & & & \\ & & & \end{bmatrix}.$$

7.3 Determinants are Really Alternating Sums

If A is an $n \times n$ matrix, then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Interpret Quantity. $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

n-routes in a directed graph

Set P.

Set N.

Correspondence.

So the determinant is the signed sum of *nonintersecting* n-routes.

Revisit our original determinants. Can we create *meaningful* directed graphs to enable our calculation of the determinants?

$$\begin{vmatrix} 2 & 5 & 14 & 23 \\ 1 & 3 & 9 & 15 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 \\ 10 & 11 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{vmatrix}$$

7.4 Working Together

$$\det\begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} \end{bmatrix} = 1 \qquad \det\begin{bmatrix} \binom{n}{0} & \binom{n+1}{0} & \binom{n+2}{0} & \binom{n+3}{0} \\ \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \binom{n+4}{1} \\ \binom{n+2}{2} & \binom{n+3}{2} & \binom{n+4}{2} & \binom{n+5}{2} \\ \binom{n+3}{3} & \binom{n+4}{3} & \binom{n+5}{3} & \binom{n+6}{3} \end{bmatrix} = 1.$$

$$\det\begin{bmatrix} \binom{n}{0} & \binom{n+1}{0} & \binom{n+2}{0} & \cdots & \binom{n+k}{0} \\ \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \cdots & \binom{n+k+1}{1} \\ \binom{n+2}{2} & \binom{n+3}{2} & \binom{n+4}{2} & \cdots & \binom{n+k+2}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{k} & \binom{n+k+1}{k} & \binom{n+k+2}{k} & \cdots & \binom{n+2k}{k} \end{bmatrix} = 1$$

$$\det\begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 2 \end{bmatrix} = n+1 \qquad \det\begin{bmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{bmatrix} = (-1)^{m+1}.$$

Assume matrix is $n \times n$.

7.5 On Your Own

1. $\det\begin{bmatrix} f_{m-r} & f_{m+s-r} \\ f_m & f_{m+s} \end{bmatrix} = (-1)^{m-r} f_{r-1} f_{s-1}$.

2. $\det\begin{bmatrix} f_m & f_{m+p} & f_{m+q} \\ f_{m+r} & f_{m+p+r} & f_{m+q+r} \\ f_{m+s} & f_{m+p+s} & f_{m+q+s} \end{bmatrix} = 0$.

3. $\det\begin{bmatrix} G_{m-1} & G_m \\ G_m & G_{m+1} \end{bmatrix} = (-1)^{m-1}(G_0G_2 - G_1^2)$.

4. For $j \geq 1$, $\det\begin{bmatrix} j & 1 & 0 & 0 & \dots & 0 \\ 1 & j & 1 & 0 & \dots & 0 \\ 0 & 1 & j & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & j & 1 \\ 0 & \dots & \dots & 0 & 1 & j \end{bmatrix} = A_n$ where $A_0 = 1$, $A_1 = j$, and for $n \geq 2$, $A_n = jA_{n-1} - A_{n-2}$. Note this also equals the alternating sum $\sum_{k=0}^{n}(-1)^k\binom{n-k}{k}j^{n-2k}$.

5. The nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of lattice paths from $(0,0)$ to (n,n) using "right" and "up" edges and staying below the line $y = x$. If

$$M_n^t = \begin{bmatrix} C_t & C_{t+1} & \cdots & C_{t+n-1} \\ C_{t+1} & C_{t+2} & \cdots & C_{t+n} \\ \vdots & & \ddots & \vdots \\ C_{t+n-1} & C_{t+n} & \cdots & C_{t+2n-2} \end{bmatrix},$$

show that $\det(M_n^0) = 1$, $\det(M_n^1) = 1$, and $\det(M_n^2) = n+1$.

6. Define

$$S_n^t = \begin{bmatrix} C_t + C_{t+1} & C_{t+1} + C_{t+2} & \dots & C_{t+n-1} + C_{t+n} \\ C_{t+1} + C_{t+2} & C_{t+2} + C_{t+3} & \dots & C_{t+n} + C_{t+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{t+n-1} + C_{t+n} & C_{t+n} + C_{t+n+1} & \dots & C_{t+2n-2} + C_{t+2n-1} \end{bmatrix}.$$

Show that $\det(S_n^0) = f_{2n}$ and $\det(S_n^1) = f_{2n+1}$.

7.6 Classic Properties of Determinants

Suppose that A, B are $n \times n$ matrices.

- The determinant changes sign when two rows are exchanged.
- If two rows of A are equal, then $\det(A) = 0$.
- A matrix with a row of zeroes has determinant equal to 0.
- $\det(A) = \det(A^T)$.
- $\det(A)\det(B) = \det(AB)$.

7.7 Vandermonde Determinant

Just as we generalized from counting tilings to summing weighted tilings, we can move from counting (nonintersecting) n-routes to summing the weights of the n-routes — just weight the edges of the directed graphs. You probably figured this out already when you tackled some of the previous determinants. This time we will weight edges with indeterminants.

Theorem 4 *The Vandermonde matrix, $V_n = [x_i^{j-1}]$ for $1 \leq i, j \leq n$, has the determinant*

$$\det V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

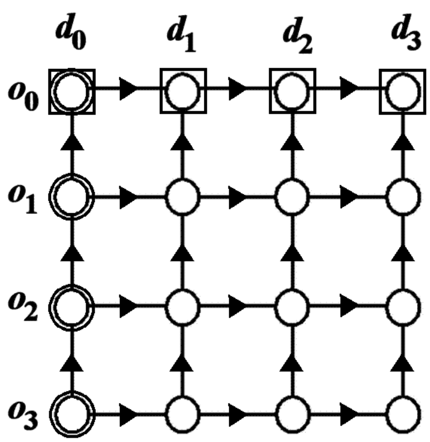
Getting a feel for the theorem. Compute

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{bmatrix}$$

We will assign weights to the $n \times n$ integer lattice so that the total weight of the paths between origin i and destination j coincide with the entry x_i^{j-1}. Then we compute weight of the nonintersecting n-route.

Concretely working with $n = 4$.

$$\det V_4 = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_n & x_n^2 & x_n^3 \end{vmatrix} = (x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$



Now extend this idea to an $n \times n$ matrix!

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