Combinatorially Thinking

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Philosophy. We want to *construct* our mathematical understanding. To this end, our goal is to situate our problems in concrete counting contexts. Most mathematicians appreciate clever combinatorial proofs. But faced with an identity, how can *you* create one?

This course will provide you with some useful combinatorial interpretations, lots of examples, and the challenge of finding your own combinatorial proofs. Throughout the next two weeks, your mantra should be to *keep it simple.*

Contents

1 Basic Tools

1.1 Some Combinatorial Interpretations

First, we need combinatorial interpretations for the objects occurring in our identities. While there are many possible interpretations, only one is presented for each mathematical object—trying, of course, to *keep it simple.*

I have included at least one method to compute each object for completeness though we will rarely rely on computation.

$n!$— *factorial*

Combinatorial: The ways to arrange numbers $1, 2, 3, \ldots, n$ in a line.

Computational: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

$\binom{n}{k}$—*binomial coefficient, n choose k*

Combinatorial: The ways to select a subset containing k elements from the set $[n] = \{1, 2, 3, \ldots, n\}$.

Computational: $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\left(\binom{n}{k}\right)$—*$n$ multichoose k*

Combinatorial: The ways to cast k votes for elements from the set $[n] = \{1, 2, 3, \ldots, n\}$.

Computational: $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$

$\left[{n \atop k}\right]$—*(unsigned) Stirling number of the first kind*

Combinatorial: The ways to arrange n people around k identical (nonempty) circular tables.

Computational: Recursively $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ and $n \geq 1$ $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$. For $k \geq 2$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n+1) \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

$\left\{ {n \atop k} \right\}$—*Stirling number of the second kind*

Combinatorial: The ways to distribute n people into k identical (nonempty) rooms.

Computational: Recursively $\left\{ {n \atop 0} \right\} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ and $n \geq 1$ $\left\{ {n \atop 1} \right\} = 1$. For $k \geq 2$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

f_n—*the nth Fibonacci number*

Combinatorial: The ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes.

Computational: $f_0 = 1, f_1 = 1$, and for $n \geq 2$ $f_n = f_{n-1} + f_{n-2}$.

$$\text{or } f_n = \frac{1}{\sqrt{5}} \left[\phi^{n+1} - \left(\frac{-1}{\phi} \right)^{n+1} \right]$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

WARNING: You might be used to the Fibonacci numbers defined in the more traditional way $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ $F_n = F_{n-1} + F_{n-2}$.

L_n—*the nth Lucas number*

Combinatorial: The ways to tile a circular $1 \times n$ board using 1×1 "squares" and 1×2 "dominoes".

Computational: $L_0 = 2, L_1 = 1$, and for $n \geq 2$ $L_n = L_{n-1} + L_{n-2}$.

$$\text{or } L_n = \phi^n + \left(\frac{-1}{\phi} \right)^n.$$

G_n—*the nth Gibonacci number*

Combinatorial: The ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes where the first tile is distinguished. There are G_1 choices for a leading square and G_0 choices for a leading domino.

Computational: G_0 and G_1 are given and for $n \geq 2$ $G_n = G_{n-1} + G_{n-2}$.

$$\text{or } G_n = \alpha\phi^n + \beta \left(\frac{-1}{\phi} \right)^n$$

where $\alpha = (G_1 + G_0/\phi)/\sqrt{5}$ and $\beta = (\phi G_0 - G_1)/\sqrt{5}$.

D_n—*the nth Derangement number*

Combinatorial: The ways to arrange $1, 2, \ldots, n$ in a line so that no number lies in its natural position.

Computational: $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$.

C_n—*the nth Catalan number*

Combinatorial: The number of lattice paths from $(0, 0)$ to (n, n) using "right" and "up" edges and staying below the line $y = x$.

Computational: $C_n = \frac{1}{n+1}\binom{2n}{n}$

$[a_0, a_1, \ldots, a_n]$—*the finite continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}$$

Combinatorial:

Numerator: The ways to tile a $1 \times n+1$ board using squares and dominoes where cell i can contain half a domino or as many as a_i squares, $0 \leq i \leq n$.

Denominator: The ways to tile a $1 \times n$ board using squares and dominoes where cell i can contain half a domino or as many as a_i squares, $1 \leq i \leq n$.

Computational: Attack with algebra to rationalize the complex fraction.

$\det(A)$—*the determinant of the $n \times n$ matrix $A = \{a_{ij}\}$.*

Combinatorial: The signed sum of nonintersecting n-routes in a directed graph with n origins, n destinations, and a_{ij} directed paths from origin i to destination j.

Computational: $\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

Example: $\det \begin{bmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{bmatrix} = 1{\cdot}8{\cdot}2 + 2{\cdot}21{\cdot}0 + 5{\cdot}5{\cdot}1 - 0{\cdot}8{\cdot}5 - 5{\cdot}2{\cdot}2 - 1{\cdot}1{\cdot}21 = 0.$

1.2 Counting Technique 1: Ask a Question and Answer Two Ways

We will use one of two techniques to *count* an identity. The first poses a question and then answers it in two different ways. One answer is the left side of the identity; the other answer is the right side. Since both answers solve the same counting question, they must be equal.

Identity 1 *For* $n \geq 1$,

$$\sum_{k=1}^{n-1} k \cdot k! = n! - 1.$$

Question: The number of ways to arrange $1, 2, 3, \ldots, n$ except for

Answer 1:

Answer 2:

Identity 2 *For* $k, n \geq 0$,

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Question: How many ways can we allocate k votes to n candidates?

Answer 1:

Answer 2:

1.3 Counting Technique 2: Description-Involution-Exception (DIE)

The second technique is to create two sets, count their sizes, and find a correspondence between them. The correspondence could be one-to-one, many-to-one, almost one-to-one, or almost many-to-one.

Identity 3 *For $k, n \geq 0$,*

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Description:

Set 1:

Set 2:

Involution:

Exception:

Identity 4 *For $n \geq 0$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Description:

Set 1:

Set 2:

Involution:

Exception:

What happens is we change the upper index of the summation to something smaller than n? larger than n?

2 Binomial Identities

2.1 Working Together

Let's use these techniques to prove some identities.

Identity 5 *For* $0 \leq k \leq n$

$$n! = \binom{n}{k} k!(n-k)!$$

Identity 6 The Binomial Theorem. *For* $n \geq 0$,

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n.$$

Identity 7 *For* $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

2.2 On Your Own

Identity 8 *For* $0 \le k \le n$, *(except* $n = k = 0$*),*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The technique above can be modified to prove:

Identity 9 *For* $n \ge 0, k \ge 0$, *(except* $n = k = 0$*),* $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right).$

Identity 10 *For* $n \ge k \ge 1$, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}.$

Identity 11 *For* $n \ge k \ge 1$, $\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}.$

Identity 12 *For* $n \ge 1$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Identity 13 *For* $n \ge 1$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Identity 14 *For* $n \ge q \ge 0$,

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

Identity 15 *For* $n \ge m, k \ge 0$,

$$\sum_{j} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}.$$

Identity 16 *For* $n \ge k \ge j \ge 0$,

$$\sum_{m} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$$

Identity 17 *For nonnegative integers* $k_1, k_2, \ldots, k_n$, *let* $N = \sum_{i=1}^{n} \binom{k_i}{2}$. *Then*

$$\sum_{1 \le i < j \le n} \binom{k_i}{2}\binom{k_j}{2} + 3\sum_{i=1} n\binom{k_i+1}{4} = \binom{N}{2}.$$

2.3 What's the Parity of $\binom{n}{k}$?

Pascal's Triangle	Serpinski-like Triangle
1	1
1 1	1 1
1 2 1	1 0 1
1 3 3 1	1 1 1 1
1 4 6 4 1	1 0 0 0 1
1 5 10 10 5 1	1 1 0 0 1 1
1 6 15 20 15 6 1	1 0 1 0 1 0 1
1 7 21 35 35 21 7 1	1 1 1 1 1 1 1 1
1 8 28 56 70 56 28 8 1	1 0 0 0 0 0 0 0 1
1 9 36 84 126 126 84 36 9 1	1 1 0 0 0 0 0 0 1 1
1 10 45 120 210 252 210 120 45 10 1	1 0 1 0 0 0 0 0 1 0 1
1 11 55 165 330 462 462 330 165 55 11 1	1 1 1 1 0 0 0 0 1 1 1 1

Theorem 1 *For $n \geq 0$, the number of odd integers in the nth row of Pascal's triangle is equal to 2^b where $b =$the number of 1s in the binary expansion of n.*

Question. How many odd numbers occur in the 76^{th} row of Pascal's triangle?

Lemma. *The parity of $\binom{n}{k}$ will be the same as the parity of the number of palindromic sequences with k ones and $n-k$ zeros.*

Description. Binary sequences with k ones and $n-k$ zeros.

Involution.

Exception.

Ans: $76 = (1001100)_2$. *Eight odd numbers.*

Count palindromic sequences with k ones and $n-k$ zeros.

$\binom{even}{odd}$

$\binom{even}{even}$

$\binom{odd}{even}$

$\binom{odd}{odd}$

Consequence. If n is even and k is odd, then $\binom{n}{k}$ is even, otherwise, $\binom{n}{k}$ has the same parity as $\binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}$ where we round $n/2$ and $k/2$ down to the nearest integer, if necessary.

Examples. Compute the parity:

$$\binom{76}{15} \qquad \binom{76}{36} \qquad \binom{76}{12}$$

Think binary!

$$\binom{(1001100)_2}{(0001111)_2} \qquad \binom{(1001100)_2}{(0100100)_2} \qquad \binom{(1001100)_2}{(0001100)_2}$$

Only way to have an odd number is if the 1s in the binary representation of k are directly below 1s in binary representation of n.

It tells us *exactly* which numbers produce odd binomial coefficients:

binary representation	k
1001100	76
1001000	72
1000100	68
1000000	64
0001100	12
0001000	8
0000100	4
0000000	0

Extension. There a similar procedure to determine the remainder of $\binom{n}{k}$ when divided by *any* prime p?

Lucas' Theorem. *For any prime p, we can determine the remainder of $\binom{n}{k}$ when divided by p from the base p expansions of n and k. If*

$$\begin{aligned} n &= b_t p^t + b_{t-1} b^{t-1} + \cdots + b_1 p^1 + b_0 \\ k &= c_t p^t + c_{t-1} b^{t-1} + \cdots + c_1 p^1 + c_0 \end{aligned}$$

then $\binom{n}{k}$ and $\binom{b_t}{c_t}\binom{b_{t-1}}{c_{t-1}} \cdots \binom{b_1}{c_1}\binom{b_0}{c_0}$ have the same remainder when divided by p.

Example. Calculate the remainder of $\binom{97}{35}$ when divided by 5.

Ans: $\binom{97}{35} \equiv \binom{3}{1}\binom{4}{2}\binom{2}{0} \equiv 3 \mod 5$

3 Fibonacci Identities

3.1 Tiling with Squares and Dominoes

Definition. Let f_n count the ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes.

Questions to explore.

1. Compute f_4, f_5, and f_6.

2. Of the square-domino tilings of the 1×6 board, how many are *breakable* after the second cell? How many have a single domino covering cells two and three?

3. For a board of length n (henceforth called an n-board),

   (a) how many start with a square?

   (b) how many start with a domino?

   (c) how many use exactly k dominoes (and what values make sense for k?)

4. What should f_0 be? What should f_{-1} be?

3.2 Working Together

Identity 18 *For* $n \geq 0$,

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_n.$$

Identity 19 *For* $n \geq 0$,

$$f_{n-1}^2 + f_n^2 = f_{2n}$$

Identity 20 *For* $n \geq 0$,

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}.$$

Identity 21 *For* $n \geq 2$,

$$3f_n = f_{n+2} + f_{n-2}.$$

3.3 On Your Own

Identity 22 *For* $n \geq 0$

$$f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

Identity 23 *For* $n \geq 0$,

$$f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}.$$

Identity 24 *For* $n \geq 0$,

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^n$$

Identity 25 *For* $n \geq 2$,

$$2f_n = f_{n+1} + f_{n-2}.$$

Identity 26 *For* $n \geq 0$,

$$\sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} = f_{2n+1}.$$

Identity 27 *For* $n \geq 0$,

$$\sum_{k=0}^{n} k f_{n-k} = f_{n+3} - (n+3)$$

Identity 28 *For* $n \geq 0$,

$$f_{2n-1} = \sum_{k \geq 0} \binom{n}{k} f_{k-1}.$$

Identity 29 *For* $n \geq 0$,

$$2^n f_{2n-1} = \sum_{k \geq 0} \binom{n}{k} f_{3k-1}.$$

Identity 30 *If* $m|n$, *then* $f_{m-1}|f_{n-1}$ *(i.e.* $F_m|F_n$*)*

Hint: If $n = qm$*, count* $(n-1)$*-tilings by considering the smallest value* j *such that the tiling is breakable at cell* $jm - 1$*. (Why must such a cell exist?)*

3.4 Combinatorial Proof of Binet's Formula

Is it possible to construct a combinatorial proof of identities involving irrational quantities? For example, using the standard Fibonacci Society definition $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, we have Binet's classic formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

A combinatorial proof is possible if we introduce *probability.* Let $\phi = \frac{1+\sqrt{5}}{2}$.

FACTS:

- $\left(\frac{1-\sqrt{5}}{2}\right) = -1/\phi$.
- $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$.
- Equivalent identity (since $F_n = f_{n-1}$).

$$f_n = \frac{1}{\sqrt{5}}\left[\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right].$$

Tile an infinite board by independently placing squares and dominoes, one after the other. At each decision, place a square with probability $1/\phi$ or a domino with probability $1/\phi^2$. The probability that a tiling begins with any particular length n sequence is $1/\phi^n$. Let q_n be the probability that an infinite tiling is breakable at cell n.

$$q_n = \tag{1}$$

Question. What is the probability that an infinite tiling is breakable at cell n?

Answer 1. q_n

Answer 2. One minus the probability that it is not breakable at cell n.

Remember $q_0 = 1$.

Unravel the recurrence in (1) to get

$$q_n = 1 - \frac{1}{\phi^2} + \frac{1}{\phi^4} - \frac{1}{\phi^6} + \cdots + \left(\frac{-1}{\phi^2}\right)^n.$$

Sum the geometric series.

Thus $f_n = \phi^n q_n =$

4 Generalizations—Lucas, Gibonacci, and Linear Recurrences

4.1 Motivating Generalization

Revisit Identity 28. *For $n \geq 0$,*

$$f_{2n-1} = \sum_{k \geq 0} \binom{n}{k} f_{k-1}.$$

Question: How many $(2n-1)$-tilings?

Answer 1: f_{2n-1}

Answer 2: Let k be the number of squares among the first n tiles.

Was the -1 really necessary in f_{2n-1} or f_{k-1} to make this argument fly?

Identity 31 *For $n \geq 0$, $p \geq -1$,*

$$f_{2n+p} = \sum_{k \geq 0} \binom{n}{k} f_{k+p}.$$

Identity seems independent of "initial conditions". Why not consider a general Fibonacci sequence (a.k.a. Gibonacci sequence)?

Definition. Let G_0 and G_1 be specified and for $n \geq 2$ define $G_n = G_{n-1} + G_{n-2}$.

The first few terms in a Gibonacci sequence are
G_0, G_1,

4.2 Weighting the Tilings

To transform the square-domino tilings of an n-board that give us the Fibonacci number f_n, to a Gibonacci number G_n we weight the tiling as follows:

weight assigned to tiling ending in a *square*:	G_1
weight assigned to tiling ending in a *domino*:	G_0

Let w_n equal the sum of the weights of the length n-tilings.
Questions to explore.

1. Compute w_1, w_2, and w_3.

2. For a board of length n (henceforth called an n-board),

   (a) what is the total weight attributable to tilings that start with a square?

   (b) what is the total weight attributable to tilings that start with a domino?

3. What should w_0 be?

So $w_n = G_n$ **for** $n \geq 0$.

Combinatorial Interpretation. For a Gibonacci sequence $G_0, G_1, G_2, \ldots$, the Gibonacci number G_n is the total weight of all square-domino tilings of length n where tilings ending withe a square have weight G_1 and tilings ending with a domino have weight G_0.

4.3 Working Together

Identity 32 *For* $m, n \geq 0$,

$$G_{m+n} = G_m f_n + G_{m-1} f_{n-1}.$$

Identity 33 *For* $n \geq 0$,

$$G_0 + G_1 + G_2 + \cdots + G_n = G_{n+2} - G_1.$$

Identity 34 *For* $n \geq 1$,

$$G_{n+1} G_{n-1} - G_n^2 = (-1)^n (G_1^2 - G_0 G_2).$$

Identity 35 *For* $n \geq 0$,

$$G_{2n} = \sum_{k \geq 0} \binom{n}{k} G_k.$$

Identity 36 *For* $n \geq 2$,

$$G_{n+2} + G_{n-2} = 3G_n.$$

4.4 On Your Own

Of special note is the **Lucas sequence**, the so-called companion sequence to the Fibonaccis. Here $L_0 = 2$, $L_1 = 1$ and for $n \geq 2$ $L_n = L_{n-1} + L_{n-2}$. While weighted tilings work fine, we could reinterpret Lucas numbers as circular tilings. (See Section 1.1.)

Identity 37 *For $n \geq 1$,*

$$G_n = G_0 f_{n-2} + G_1 f_{n-1}.$$

Identity 38 *For $n \geq 0$,*

$$G_1 + \sum_{k=1}^{n} G_{2k} = G_{2n+1}.$$

Identity 39 *For $n \geq 0$,*

$$G_0 G_1 + \sum_{k=1}^{n} G_k^2 = G_n G_{n+1}.$$

Identity 40 *For $n \geq 0$,*

$$5f_n = L_n + L_{n+2}$$

Identity 41 *For $n \geq 0$,*

$$L_n^2 = L_{2n} + (-1)^n \cdot 2.$$

Identity 42 *Let $G_0, G_1, G_2, \ldots$ and $H_0, H_1, H_2, \ldots$ be Gibonacci sequences. Then for $0 \leq m \leq n$,*

$$G_m H_n - G_n H_m = (-1)^m (G_0 H_{n-m} - G_{n-m} H_0).$$

Identity 43 *For $n \geq 0$,*

$$2^n G_{2n} = \sum_{k \geq 0} \binom{n}{k} G_{3k}.$$

4.5 Weighting Tiles Individually

For the Gibonacci numbers, we weighted the tiling based on the last tile used (G_0 if it ended in a domino and G_1 if it ended in a square.) We have seen, that sometimes it was simpler to think of the weight as being attached to the last tile itself. We could do this if the rest of the board had "weight 1" and the weight obtained by concatenating two boards is the **product** of the weights of its parts. This leads to the following idea:

Definition. The *weight* of a tiling T, denoted $w(T)$, is the product of the weight of the individual tiles. The *total weight* for tilings of length n, denoted t_n, is the sum of the weights of all tilings of length n.

Example. Suppose we tiling n-boards with squares of weight s and dominoes of weight d. Find t_1, t_2, t_3, t_4.

Can you find a recurrence to express t_n?

What are the initial conditions for the situation described above?

For general linear recurrences of order 2

Combinatorial Interpretation. Let s, d, a_0, a_1 be given and for $n \geq 2$, define

$$a_n = sa_{n-1} + da_{n-2}.$$

For $n \geq 1$, a_n is the total weight of length n-tilings created from weight s squares and weight d dominoes except for the last tile, which may be a weight a_1 square or a weight da_0 domino.

For the three identities below, we will assume ideal initial conditions: suppose $a_0 = 1, a_1 = s$ and for $n \geq 2$, $a_n = sa_{n-1} + ta_{n-2}$.

Identity 44 *For $m, n \geq 1$,*

$$a_{m+n} = a_m a_n + t a_{m-1} a_{n-1}.$$

Identity 45 *For $n \geq 2$,*

$$a_n - 1 = (s-1)a_{n-1} + (s+t-1) \sum_{k=1}^{n-1} a_{k-1}.$$

Identity 46 *For any $1 \leq c \leq s$, for $n \geq 0$,*

$$a_n - c^n = (s-c)a_{n-1} + ((s-c)c + t) \sum_{k=1}^{n-1} a_{k-1} c^{n-1-k}.$$

4.6 A Gibonacci Magic Trick

Secretly write a positive integer in row 1 and another positive integer in row 2.

row	integers
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Next add those numbers together and put the sum in row 3. Add row 2 and row 3 and place the answer in row 4. Continue in this fashion until numbers are in rows 1 through 10. Now using a calculator, if you wish, add all the numbers in rows 1 through 10 together.

While the volunteer is adding, the mathemagician glances at the sheet of numbers for just a second and instantly reveals the sum. How?

Suppose $G_0 = x$ and $G_1 = y$ are our secret numbers....

row	integers
1	x
2	y
3	
4	
5	
6	
7	
8	
9	
10	

As a final flourish to the mathemagician's performance he adds, "Now using a calculator, divide the number in row 10 by the number in row 9 and announce the first three digits of your answer. What's that you say? 1.61? Now turn over the paper and look what I have written." The back of the paper says, "I predict the number 1.61."

Why does the ratio work?

> For any two fractions $\frac{a}{b} < \frac{c}{d}$ with positive numerators and denominators, it is easy to show that
>
> $$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

What are the implications of this for (row 10)/(row 9)?

5 Continued Fractions

5.1 Introductions and Explorations

Definition. Given integers $a_0 \geq 0$, $a_1, \leq 1, a_2 \geq 1, \ldots, a_n \geq 1$, define *finite continued fraction*, denoted $[a_0, a_1, \ldots, a_n]$, to be the fraction in lowest terms for

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Compute: $[3, 4, 5, 2]$ and $[2, 5, 4, 3]$

Find finite continued fractions that represent 37/17 and 9/5.

Understanding Continued Fractions Through Algebra

Suppose $[a_0, a_1, \ldots, a_n] = \dfrac{p(a_0, a_1, \ldots, a_n)}{q(a_0, a_1, \ldots, a_n)}$. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = a_0 + \frac{1}{\frac{p(a_1, \ldots, a_n)}{q(a_1, \ldots, a_n)}}$$

Initial conditions? $p(a_0)$ $q(a_0)$ $p(a_0, a_1)$ $q(a_0, a_1)$

This yields the following relations for the numerator and denominator of finite continued fractions:

$$\begin{aligned} p(a_0, a_1, \ldots, a_n) &= a_0 p(a_1, \ldots, a_n) + p(a_2, \ldots, a_n) \\ q(a_0, a_1, \ldots, a_n) &= p(a_1, \ldots, a_n) \end{aligned}$$

Counting Context

Let $K(a_i, a_{i+1}, \ldots, a_j)$ be the total weight of the square-domino tilings of an $1 \times (j-i+1)$-board (with cell indexed from i to j) where dominoes have weight 1 and a square on cell k has weight a_k, $i \leq k \leq j$.

Compute $K(3, 4, 5, 2)$.

Find a recurrence for $K(a_0, a_1, a_2, \ldots, a_n)$ based on the first tile.

Initial conditions:

$K(a_0) =$

$K(a_0, a_1) =$

Combinatorial Consequence. The finite partial fraction

$$[a_0, a_1, \ldots, a_n] = \frac{K(a_0, a_1, \ldots, a_n)}{K(a_1, \ldots, a_n)}.$$

5.2 Working Together

Theorem 2 *Reversing the entries in a finite continued fraction does not change the value of the numerator.*

Definition. An *infinite continued fraction* $[a_0, a_1, a_2, \ldots] = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n]$. The finite approximation $[a_0, a_1, \ldots, a_n] = r_n = \frac{p_n}{q_n}$ is called a *convergent* of the continued fraction.

Identity 47 *The difference between consecutive convergents of* $[a_0, a_1, \ldots]$ *is:*

$$r_n - r_{n-1} = \frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

Equivalently, after multiplying both sides by $q_n q_{n-1}$, *we have*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

One consequence is that p_n/q_n *is in lowest terms since we have an integer combination of* p_n *and* q_n *that equals* ± 1.

Identity 48 *The difference between every other convergent of* $[a_0, a_1, \ldots]$ *is:*

$$r_n - r_{n-2} = \frac{(-1)^n a_n}{q_n q_{n-2}}.$$

Equivalently, after multiplying both sides by $q_n q_{n-2}$*, we have*

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

How can you use the last two identities to see that infinite continued fractions always converge (and to an irrational number at that!)

Identity 49 $[a_0, a_1, \ldots, a_{n-1}, 1] = [a_0, a_1, \ldots, a_{n-1} + 1]$

5.3 On Your Own

Prove the following by direct combinatorial argument.

Identity 50 *For* $n \geq 0$, $[a_0, a_1, \ldots, a_n] = [1, 1, \ldots, 1] = \frac{f_{n+1}}{f_n}$

Identity 51 *For* $n \geq 0$, $[a_0, a_1, a_2, \ldots, a_n] = [3, 1, 1, \ldots, 1] = L_{n+2}/f_n$.

Identity 52 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [1, 1, \ldots, 1, 3] = L_{n+2}/L_{n+1}$.

Identity 53 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [4, 4, \ldots, 4, 3] = f_{3n+3}/f_{3n}$.

Identity 54 *For* $n \geq 1$, *let* $[a_0, a_1, a_2, \ldots, a_n] = [4, 4, \ldots, 4, 4] = f_{3n+5}/f_{3n+2}$.

Problem. Use Identities 47 and 48 to show that infinite continued fractions are well defined (i.e. the defining limit actually exists).

Problem. If an infinite continued fraction $[a_0, a_1, \ldots]$ converges to r, can you show that r must be irrational?

> *Hint: Show that* $0 < |r - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$. *Then assume that* r *is rational to arrive at a contradiction.*

5.4 Primes of form $4m + 1$

Theorem 3 *Any prime of the form $4m + 1$ can be (uniquely) written as the sum of the squares of two positive integers.*

Examples. Write 5, 13, and 17 as the sum of two squares of two positive integers.

Proof. Suppose that $p = 4m + 1$ is prime and consider the continued fraction expansions of

$$\frac{p}{1}, \frac{p}{2}, \ldots, \frac{p}{2m}.$$

6 Alternating Sums

6.1 Working Together

Alternating sums (or sums where the sign of the summand alternates between positive and negative), are almost exclusively counted using the D.I.E. Method. The goal is to describe two sets—one that is created from the positive "stuff" and the second from the negative "stuff", combinatorially interpret the quantity that changes sign, pair of positive and negative "stuff", and count the exceptions. The first example of this technique was proving Identity 4 and its generalization

Identity 55 Generalization of Identity 4. *For $m \geq 0$ and $n > 0$*

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}.$$

.

Identity 56 *For $n \geq 0$,*

$$\sum_{k=0}^{n}(-1)^k f_k = 1 + (-1)^n f_{n-1}.$$

Note $f_{-1} = 0$.

Interpret Quantity. f_k

Set P.

Set N.

Correspondence.

Exceptions.

Can you figure out the Gibonacci version of this identity?

Identity 57 *For $n \geq 2$,*

$$\sum_{k=0}^{n} (-1)^k k \binom{n}{k} = 0.$$

Interpret Quantity. $k\binom{n}{k}$

Set P**.**

Set N**.**

Correspondence.

Exceptions.

Identity 58

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

Interpret Quantity. $\frac{n!}{k!}$

Set P**.**

Set N**.**

Correspondence.

Exceptions.

6.2 On Your Own

Identity 59 *For $0 \leq m < n$,*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m}(-1)^k = 0.$$

Identity 60 *For $n \geq 0$,* $\sum_{k\geq 0}(-1)^k\binom{n-k}{k} = \begin{cases} 1 & \textit{if } n \equiv 0 \textit{ or } 1 (\textit{mod } 6) \\ 0 & \textit{if } n \equiv 2 \textit{ or } 5 (\textit{mod } 6) \\ -1 & \textit{if } n \equiv 3 \textit{ or } 4 (\textit{mod } 6) \end{cases}$

Identity 61 *For $n \geq 0$,* $\sum_{k\geq 0}(-1)^k\binom{n-k}{k}2^{n-2k} = n+1.$

Identity 62

$$\sum_{k\geq 0}(-1)^k\binom{n-k}{k}(xy)^k(x+y)^{n-2k} = \sum_{j\geq 0} x^{n-j}y^j$$

Identity 63 *For $n \geq 0$,* $\sum_{k=0}^{n}(-1)^k f_{2k} = (-1)^n f_n^2.$

Identity 64 *For $n \geq 3$,*

$$\sum_{i=0}^{n}(-1)^i i f_i = (-1)^n(nf_{n-1} + f_{n-3}) + 1$$

Identity 65 *For $0 \leq k \leq n$*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k = n! \cdot \delta_{k,n}$$

6.3 Extend and Generalize by Playing with Parameters

Identity 66

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1} = 0.$$

Interpret Quantity. $\binom{n}{r}\binom{2n-2r}{n-1}$

Set P.

Set N.

Correspondence.

Exceptions.

Questions to consider for generalizations

- What role does $n-1$ play in $\binom{2n-2r}{n-1}$?
- What can you say if you replace $n-1$ with an arbitrary $m < n$?
- What can you say if you replace $n-1$ with an arbitrary $m > n$?
- Rather than considering n pairs, what is we painted n triples? quadruples? k-tuples?
- Does the summation have to go all the way from $r = 0$ to $r = n$? Can we compute a partial sum...say to some fixed $s < n$?

Identity 67 *For* $0 \le m < n$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{m} = 0.$$

Identity 68 *For* $n, m \ge 0$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{m} = 2^{2n-m}\binom{n}{m-n}.$$

Identity 69 *For* $0 \le m < n$ *and* $k \ge 1$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{kn-kr}{m} = 0.$$

Identity 70

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{kn-kr}{n} = k^n.$$

Identity 71

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{kn-kr}{n+1} = nk^{n-1}\binom{k}{2}.$$

Identity 72 *For* $0 \le m < s \le n$,

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{m} = 0.$$

Identity 73 *For* $0 \le s \le n$,

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{s} = 2^s.$$

6.4 Beyond DIE—a.k.a. "What happens after you DIE?"

This identity requires the twist of iterated involutions.

Identity 74

$$\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{n-1}{j-1}\binom{n-1}{k-1}\binom{j+k}{j} = 2.$$

Interpret Quantity. $\binom{n-1}{j-1}\binom{n-1}{k-1}\binom{j+k}{j}$

Set P.

Set N.

Correspondence.

In the preceding analysis, elements 1 and $n+1$ represented the guaranteed members of X and Y respectively. There is no reason to believe we are restricted to specifying only one member of each set. Why not two? three? Why not specify a members of X and b members of Y? Originally, X and Y were selected from disjoint sets of size n. Did they have to be the same size? Why not choose X from an n-set and Y from an m-set? With very little additional effort, you can modify the above description, involutions, and exceptions to obtain the following generalization:

Identity 75

$$\sum_{j=a}^{n}\sum_{k=b}^{m}(-1)^{j+k}\binom{n-a}{j-a}\binom{m-b}{k-b}\binom{j+k}{j} = \binom{a+b}{n-m+b}.$$

6.5 Binet Revisited

You are now prepared for one last attack of Binet's formula through finite weighted colored tilings. If $\phi = \frac{1+\sqrt{5}}{2}$, then $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ and Binet's formula can be expressed as

$$f_n = \frac{1}{\sqrt{5}}(\phi^{n+1} - \overline{\phi}^{n+1}).$$

Definition. Let B_n be the total weight of a square domino tiling of a $1 \times n$ board where weights of tiles are assigned as follows:

tile type	*tile location*	*weight assigned*
domino	anywhere	1
white square	any cell ≥ 2	ϕ
white square	cell 1	$\frac{\phi^2}{\sqrt{5}}$
black square	any cell ≥ 2	$\overline{\phi}$
black square	cell 1	$\frac{-\overline{\phi}^2}{\sqrt{5}}$

Compute B_0, B_1, B_2

For $n \geq 2$, find a recurrence for B_n based on the weight of the last tile.

So $B_n = f_n$.

"Involution" Gather tilings that add to zero:

Exceptions

7 Determinants

Calculate the following determinants:

$$\begin{vmatrix} 1 & 1 \\ 10 & 11 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{vmatrix} =$$

In general

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

What about

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

7.1 The Big Formula

The determinant of an $n \times n$ matrix $A = \{a_{ij}\}$ is

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

A permutation σ of S_n is an arrangement of $[n]$ (there are $n!$ of these). In terms of a matrix, it corresponds to an $n \times n$ matrix of 1s and 0s where there is exactly one 1 in each row and column.

Example. Find the 6×6 matrices corresponding to the following permutations.

123456 145236 534621

The sign(σ) is ± 1 depending on *parity* of the number of row exchanges (transpositions) needed to transform it to the identity. (Even $\to +1$; Odd $\to -1$.)

Permutations of S_3 For each permutation matrix below determine the corresponding permutation of [3] and the sign of the permutation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

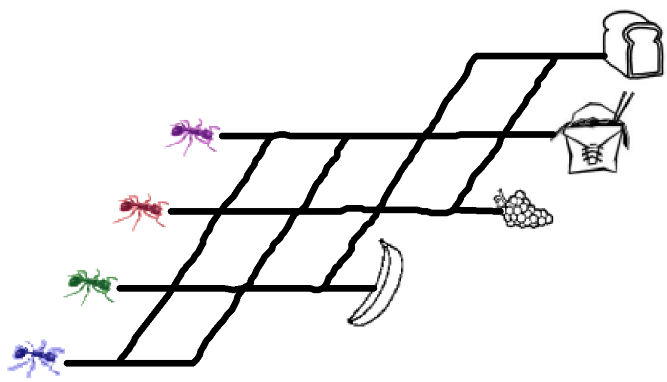
Check your Understanding. Use the big formula to compute

$$\det(A) = \begin{vmatrix} 2 & 5 & 14 & 23 \\ 1 & 3 & 9 & 15 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{vmatrix} =$$

This is a nice answer. Our goal will be to "see" why.

7.2 Matrices from Determined Ants

Given a directed graph with n origins (the ants) and n destinations (the food)



create a matrix A where entry a_{ij} represents the number of paths for ant i to get to food j.

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} & & 14 & 23 \\ & & 9 & 15 \\ & & & \\ & & & \end{bmatrix}.$$

7.3 Determinants are Really Alternating Sums

If A is an $n \times n$ matrix, then

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Interpret Quantity. $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

n-routes in a directed graph

Set P.

Set N.

Correspondence.

So the determinant is the signed sum of *nonintersecting* n-routes.

Revisit our original determinants. Can we create *meaningful* directed graphs to enable our calculation of the determinants?

$$\begin{vmatrix} 2 & 5 & 14 & 23 \\ 1 & 3 & 9 & 15 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 \\ 10 & 11 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 5 \\ 5 & 8 & 21 \\ 0 & 1 & 2 \end{vmatrix}$$

7.4 Working Together

$$\det\begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} \end{bmatrix} = 1$$

$$\det\begin{bmatrix} \binom{n}{0} & \binom{n+1}{0} & \binom{n+2}{0} & \binom{n+3}{0} \\ \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \binom{n+4}{1} \\ \binom{n+2}{2} & \binom{n+3}{2} & \binom{n+4}{2} & \binom{n+5}{2} \\ \binom{n+3}{3} & \binom{n+4}{3} & \binom{n+5}{3} & \binom{n+6}{3} \end{bmatrix} = 1.$$

$$\det\begin{bmatrix} \binom{n}{0} & \binom{n+1}{0} & \binom{n+2}{0} & \cdots & \binom{n+k}{0} \\ \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \cdots & \binom{n+k+1}{1} \\ \binom{n+2}{2} & \binom{n+3}{2} & \binom{n+4}{2} & \cdots & \binom{n+k+2}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{k} & \binom{n+k+1}{k} & \binom{n+k+2}{k} & \cdots & \binom{n+2k}{k} \end{bmatrix} = 1$$

$$\det\begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 2 \end{bmatrix} = n+1$$

Assume matrix is $n \times n$.

$$\det\begin{bmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{bmatrix} = (-1)^{m+1}.$$

7.5 On Your Own

1. $\det \begin{bmatrix} f_{m-r} & f_{m+s-r} \\ f_m & f_{m+s} \end{bmatrix} = (-1)^{m-r} f_{r-1} f_{s-1}.$

2. $\det \begin{bmatrix} f_m & f_{m+p} & f_{m+q} \\ f_{m+r} & f_{m+p+r} & f_{m+q+r} \\ f_{m+s} & f_{m+p+s} & f_{m+q+s} \end{bmatrix} = 0.$

3. $\det \begin{bmatrix} G_{m-1} & G_m \\ G_m & G_{m+1} \end{bmatrix} = (-1)^{m-1}(G_0 G_2 - G_1^2).$

4. For $j \geq 1$, $\det \begin{bmatrix} j & 1 & 0 & 0 & \dots & 0 \\ 1 & j & 1 & 0 & \dots & 0 \\ 0 & 1 & j & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & j & 1 \\ 0 & \dots & \dots & 0 & 1 & j \end{bmatrix} = A_n$ where $A_0 = 1$, $A_1 = j$, and for $n \geq 2$, $A_n = jA_{n-1} - A_{n-2}$. Note this also equals the alternating sum $\sum_{k=0}^{n} (-1)^k \binom{n-k}{k} j^{n-2k}$.

5. The nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of lattice paths from $(0,0)$ to (n,n) using "right" and "up" edges and staying below the line $y = x$. If

$$M_n^t = \begin{bmatrix} C_t & C_{t+1} & \cdots & C_{t+n-1} \\ C_{t+1} & C_{t+2} & \cdots & C_{t+n} \\ \vdots & & \ddots & \vdots \\ C_{t+n-1} & C_{t+n} & \cdots & C_{t+2n-2} \end{bmatrix},$$

show that $\det(M_n^0) = 1$, $\det(M_n^1) = 1$, and $\det(M_n^2) = n + 1$.

6. Define

$$S_n^t = \begin{bmatrix} C_t + C_{t+1} & C_{t+1} + C_{t+2} & \dots & C_{t+n-1} + C_{t+n} \\ C_{t+1} + C_{t+2} & C_{t+2} + C_{t+3} & \dots & C_{t+n} + C_{t+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{t+n-1} + C_{t+n} & C_{t+n} + C_{t+n+1} & \dots & C_{t+2n-2} + C_{t+2n-1} \end{bmatrix}.$$

Show that $\det(S_n^0) = f_{2n}$ and $\det(S_n^1) = f_{2n+1}$.

7.6 Classic Properties of Determinants

Suppose that A, B are $n \times n$ matrices.

- The determinant changes sign when two rows are exchanged.

- If two rows of A are equal, then $\det(A) = 0$.

- A matrix with a row of zeroes has determinant equal to 0.

- $\det(A) = \det(A^T)$.

- $\det(A)\det(B) = \det(AB)$.

7.7 Vandermonde Determinant

Just as we generalized from counting tilings to summing weighted tilings, we can move from counting (nonintersecting) n-routes to summing the weights of the n-routes — just weight the edges of the directed graphs. You probably figured this out already when you tackled some of the previous determinants. This time we will weight edges with indeterminants.

Theorem 4 *The Vandermonde matrix, $V_n = [x_i^{j-1}]$ for $1 \le i, j \le n$, has the determinant*

$$\det V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

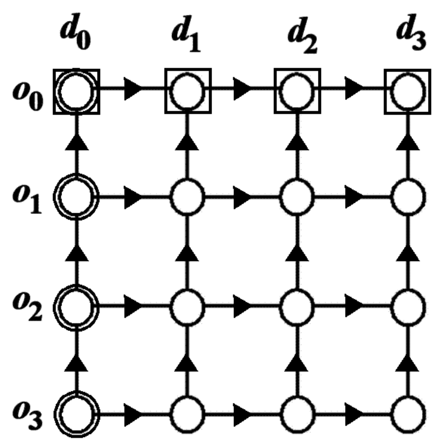
Getting a feel for the theorem. Compute

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{bmatrix}$$

We will assign weights to the $n \times n$ integer lattice so that the total weight of the paths between origin i and destination j coincide with the entry x_i^{j-1}. Then we compute weight of the nonintersecting n-route.

Concretely working with $n = 4$.

$$\det V_4 = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_n & x_n^2 & x_n^3 \end{vmatrix} = (x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$



Now extend this idea to an $n \times n$ matrix!

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For future questions, identity challenges, and sharing your successes, I'd love to hear from you at `jjquinn@u.washington.edu`.