
CALCULUS & ANALYTIC GEOMETRY I

Limits Algebraically

Big Idea On *nice* functions, limits behave properly. (By that, I mean you can manipulate the symbol “lim” the way you would expect algebraically.)

The Laws. Suppose that c is a constant, n an integer, and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

SUM $\lim_{x \rightarrow a} [f(x) + g(x)] =$

DIFFERENCE $\lim_{x \rightarrow a} [f(x) - g(x)] =$

CONSTANT MULTIPLE $\lim_{x \rightarrow a} [cf(x)] =$

PRODUCT $\lim_{x \rightarrow a} [f(x)g(x)] =$

QUOTIENT $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} =$

POWER $\lim_{x \rightarrow a} [f(x)]^n =$

ROOT $\lim_{x \rightarrow a} [f(x)]^{1/n} =$

$$\lim_{x \rightarrow a} c =$$

$$\lim_{x \rightarrow a} x =$$

Problem. Use the limit laws to evaluate:

$$\lim_{x \rightarrow -1} 3(x^2 + 1)^4$$

$$\lim_{x \rightarrow 4^-} \sqrt{16 - x^2}$$

Other extremely useful properties.

- **Direct Substitution Property.** If f is a polynomial or rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

- If two functions are equal *except* at a fixed point a , their limits (if they exist) as they approach a must be equal.

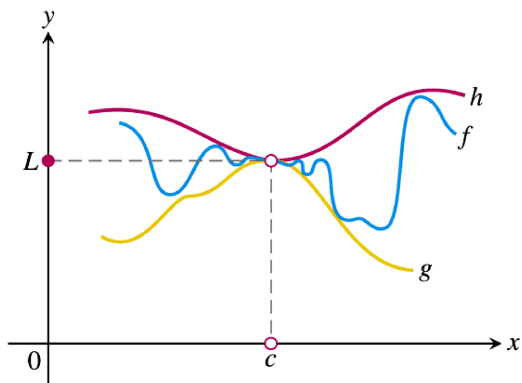
$$\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1}$$

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

- **The Sandwich Theorem.** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L$.



Example. Evaluate $\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x}$

To think about over the break. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$ hold for all values of x close to 0. What does this tell you about $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$?

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Continuity

Nice functions are continuous....

Definition. A function $f(x)$ is *continuous at a(n interior) point c* , if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

A function is *continuous at a left endpoint a* or is *continuous at a right endpoint b* if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b).$$

How can a function fail to be continuous at a point?

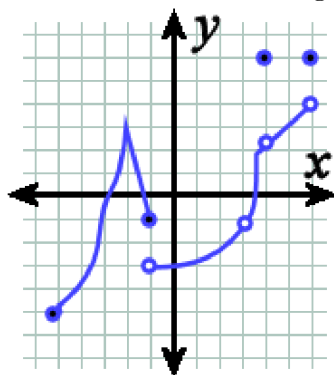
Definition. A function is called *continuous* if it is continuous at every point of its domain.

Is $f(x) = 1/x^2$ continuous?

Continuous functions include ...

Theorem 7, pg 124

Where does the following function fail to be continuous on the interval $[-5, 6]$?



Good News—Algebraic combinations of continuous functions are continuous (whenever they are defined). So if f and g are continuous at $x = c$, then so are $f + g$, $f - g$, $f \cdot g$, $k \cdot f$ for a constant k , f/g provided $g(c) \neq 0$, and $f^{r/s}$ provided it is defined and r and s are integers. Inverses of continuous functions are continuous. Compositions of continuous functions are continuous.

Problems. For what real numbers do these functions fail to be continuous?

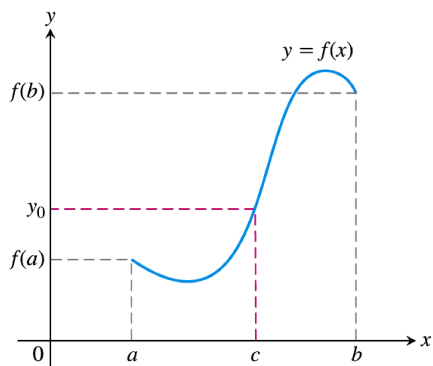
$$y = \frac{1}{x-2} - 3x$$

$$y = \frac{1}{|x|+1} - \frac{x^2}{2}$$

$$y = \frac{\sin x}{x}$$

$$y = \sqrt{2x+3}$$

Intermediate Value Theorem for Continuous Functions A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. Said another way, given y_0 between $f(a)$ and $f(b)$...



Problem. Prove that there is a root of the equation $x^4 + x = 3$ in the interval $(1, 2)$.