Calculus & Analytic Geometry I

Related Rates

Related Rates—word problems based on the chain rule. The goal is to find a rate of change from other known rates.

Strategy.

- Draw a picture and label the variables and constants.
- Write down the given information.
- Write down what you are asked to find.
- Write an equation that relates the variables.
- Differentiate with respect to t.
- Evaluate using known values to find unknown rate.

Problems

1. Suppose that the edge length x, y, and z of a closed rectangular box are changing at the following rates:
   $\frac{dx}{dt} = 1$ m/s $\qquad \frac{dy}{dt} = -2$ m/s $\qquad \frac{dz}{dt} = 1$ m/s.
   Find the rates at which the box's area, surface area, and diagonal length are changing when $x = 4$, $y = 3$, and $z = 2$.

2. A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 3 ft/sec. How fast is the top of the ladder sliding down the wall then?

At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

At what rate is the angle θ between the ladder and the ground changing then?

3. A man 6 ft tall walks at a rate of 5 ft/sec toward a street light that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the light?

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Linearization (A.K.A. Linear Approximation)

Linear Approximations. If a function f is "nice" at a point a, we can approximate the function near a by the equation of the line through $(a, f(a))$ with slope $f'(a)$. In other words, we use the equation of the tangent line to approximate the function near a.

> **Problem.** Make a rough sketch of the graph $f(x) = \sqrt{x}$. Find the linearization of $f(x)$ at the point $(4, 2)$.
>
> Estimate $\sqrt{4.0036}$.
>
> How close to the true value of $\sqrt{4.0036}$ is our estimate using the linear approximation?

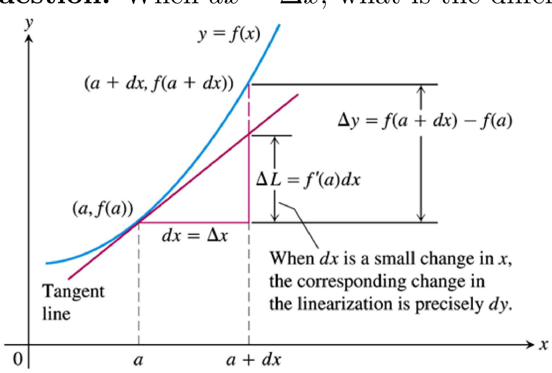
Differentials. Let $y = f(x)$ be a differentiable function. The **differential** dx is an *independent variable.* The **differential** dy is defined as

$$dy = f'(x)dx.$$

Problem

1. Find dy if $f(x) = \sqrt{x}$.
2. Find the value of dy when $x = 4$ and $dx = .0036$.

Question. When $dx = \Delta x$, what is the difference between Δy and dy?



So the linearization uses dy to approximate the true change Δy. Said another way...

$$f(a+\Delta x) = f(a) + \Delta y \approx f(a) + df.$$

Problems. Find dy for $y = (1+x)^k$ where k is a constant. Then estimate $(1.002)^{50}$ and $\sqrt[3]{1.009}$.

We can use differentials to analyze how error propagates through our calculations. Suppose we are calculating the area of a square by measuring the length of one of its sides and then using the formula $A = s^2$. Find dA when $s = 10$ cm.

If our measurement of the side is accurate to within 1 mm, how accurate is our calculation for the area?

The dA calculated above is the absolute error.
The *relative error* compares the absolute error to the calculated value.

The relative error in the measurement is $\dfrac{\Delta s}{s} = \dfrac{ds}{s} =$

The relative error in the area calculation is $\dfrac{dA}{A} =$

Local Approximation by Polynomials. For a function $f(x)$ the linear approximation of f near $x = a$ is the line

$$y = f(a) + f'(a)(x-a).$$

Find $y(a)$ and $y'(a)$.

A better approximation of $f(x)$ is a polynomial of higher order which exactly matches the values of the function and some more of its derivatives at the point $x = a$. These are called *Taylor polynomials.* The **nth-degree Taylor polynomial of f centered at a** is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$