Calculus & Analytic Geometry I

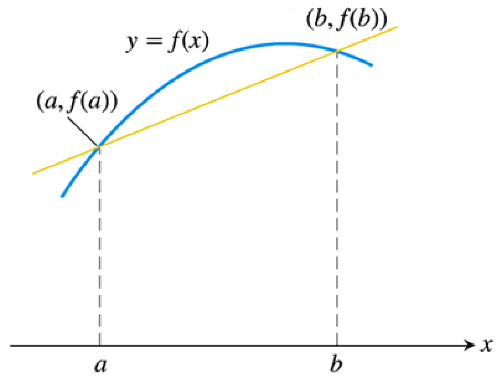
The Theorems—Most Especially the MVT

The Intermediate Value Theorem. A function $y = f(x)$ that is continuous on $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Rolle's Theorem. If a function $y = f(x)$ is

- continuous on $[a, b]$
- differentiable on (a, b)
- $f(a) = f(b)$

then there exists a c in (a, b) such that $f'(c) = 0$.



The Mean Value Theorem. If a function $y = f(x)$ is

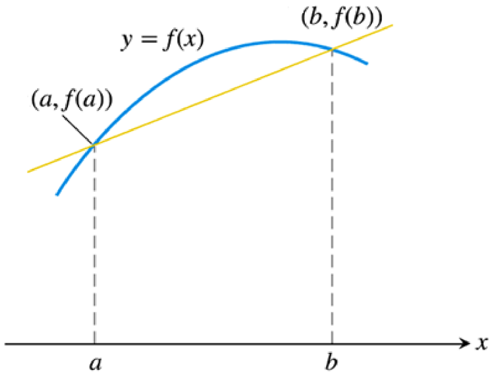
- continuous on $[a, b]$
- differentiable on (a, b)

then there exists a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof of Rolle's Theorem.

Application of Rolle's Theorem. Show that the function $f(x) = \frac{1}{1-t} + \sqrt{1+t} - 3.1$ has exactly one real zero in the interval $(-1, 1)$.

Proof of the MVT.



Let $h(x) = f(x) -$ (equation of secant line).

Corollary 1. If $f'(x) = 0$ at each point of (a, b), then $f(x)$ is a constant function on the interval (a, b). (i.e. $f(x) = C$ for some real number C.)

Corollary 2. If $f'(x) = g'(x)$ at each point of (a, b), then $f(x)$ and $g(x)$ differ by a constant function on the interval (a, b). (i.e. $f(x) = g(x) + C$ for some real number C.)

Problems.

1. Find all possible functions $f(x)$ such that $f'(x) = x^3$.
2. Find the function $g(t)$ such that $g'(t) = e^{2t}$ and $g(0) = \frac{3}{2}$.
3. Suppose the acceleration of a body is measure as 9.8 m/s^2 and we know that $v(0) = -3$ m/s, $s(0) = 5$ m. Find an equation to describe the body's position at time t.

The Derivative Tests

Question. How do you know where a function is increasing or decreasing?

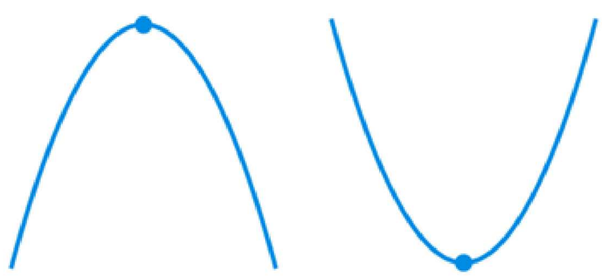
Question. How can a function behave at a transition point as it changes from increasing to decreasing or vice versa?

Question. How does this help us identity local extrema?

Consider a function whose derivative is $f'(x) = x^{-1/2}(x-3)$. Where is the function increasing? decreasing? Identify inputs that give local extremes.

Knowing how quickly the derivative of a function is changing can also give important information about local extremes.

Definition. The graph of a differentiable function $y = f(x)$ is *concave up* on an open interval I if f' is increasing on I and *concave down* if f' is decreasing on I.



Definition. A point where the graph of a function has a tangent line and where the concavity changes is called a *point of inflection*.

First Derivative Test. Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly c itself. Moving across c from left to right

- if f' changes from negative to positive at c, the f has a local ______________ at c;
- if f' changes from positive to negative at c, the f has a local ______________ at c;
- if f' does not change sign at c, then f has no local extremum at c.

Second Derivative Test. Suppose f'' is continuous on an open interval that contains $x = v$.

- If $f'(c) = 0$ and $f'' < 0$, then f has a local ______________ at $x = c$;
- If $f'(c) = 0$ and $f'' > 0$, then f has a local ______________ at $x = c$;
- If $f'(c) = 0$ and $f'' = 0$, then the test fails. The function f may have a local maximum, a local minimum or neither.

The rest of today and tomorrow, we will focus on sketching curves using these two tests.

Problem. Sketch the general shape of a curve satisfying the given information

interval:	$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
sign of f' :	$-$	$-$	$-$	$+$
sign of f'' :	$+$	$-$	$+$	$+$

Problem. Find all local extrema of $f(x) = -2\cos x - \cos^2 x$ on the interval $-\pi \leq x \leq \pi$.