Calculus & Analytic Geometry I

The Fundamental Theorem of Calculus

Warm-up. What does definite integral $\int_a^b f(x)dx$ represent?

Compute $\int_1^b x^2 dx$ by taking the limit of a lower Riemann Sum.

What if $f(x)$ is negative?

Compute $\int_2^4 (1-x)dx$

So a definite integral represents a *signed* area—

- where $f(x)$ is above the x-axis, the definite integral is the area
- where $f(x)$ is below the x-axis, the definite integral is the negative of the area.

Compute $\int_0^{2\pi} \sin(x)dx$.

Properties of Definite Integrals

1. Order of Integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. Zero Width Integral: $\int_a^a f(x)dx =$

3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any number k

   $\int_a^b -f(x)dx = -\int_a^b f(x)dx$

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int$

6. Max-Min Inequality: If f attains a maximum and minimum value on the interval $[a, b]$ then

$$\min f \cdot (b-a) \leq \int_a^b f(x)dx \leq \max f \cdot (b-a)$$

7. Domination: If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Suppose that $\int_1^2 f(x)dx = -4$, $\int_1^5 f(x)dx = 6$, and $\int_1^5 g(x)dx = 8$. Find

$\int_2^5 f(x)dx$ $\int_1^5 [4f(x) - g(x)]dx$ $\int_2^2 f(x)dx$ $\int_5^1 [g(x) - f(x)]dx$

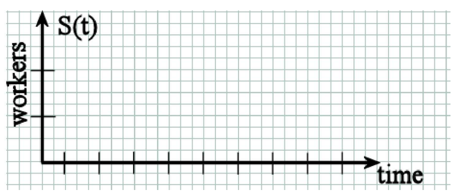
Another Acculmulation Problem. Four students are painting a house in shifts. The hours worked are shown below:

Worker	begin	end	hours worked
Chris	9 am	12 pm	
Toni	12 pm	4 pm	
Sam	10 am	2 pm	
Jo	2 pm	5 pm	

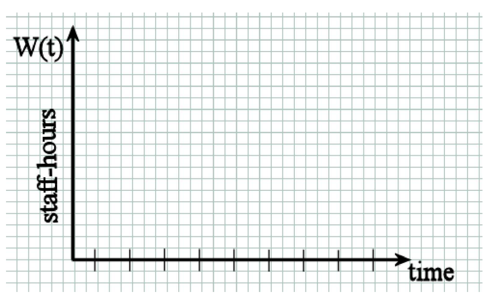
1. How many hours does each person work?

2. As the workers go through the day, they put in one token for each hour worked. How many tokens are there total at 5:00 pm? What does this represent?

3. Let $S(t)$ represent the number of people working at time t. Graph $S(t)$ verses time (9 to 5). What does the "area under this graph" represent?



4. Let $W(t)$ represent the work (or staff-hours) accumulated from 9 am until time t. Graph the function $W(t)$ versus time.



5. What is the relationship between the graph in part (3) and (4)?

Main Event. *Fundamental Theorem of Calculus, Part I.* If f is continuous on $[a, b]$, then its accumulation function $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b). Further more its derivative

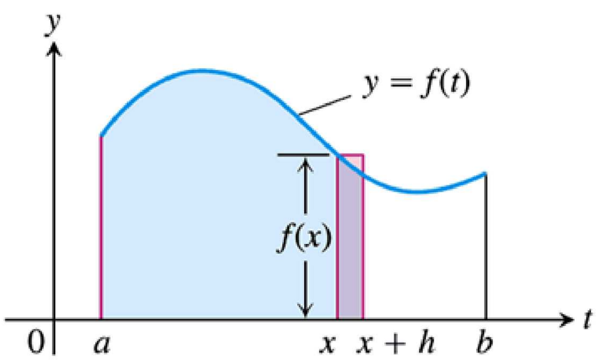
$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Illustrations. Find $\frac{dy}{dx}$

$$y = \int_0^x \cos t dt \qquad y = \int_\pi^x \cos t dt \qquad y = \int_0^{e^x} \cos t dt$$

Why? Let's interpret the difference quotient.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$



What is the difference between

$$\int f(t)dt \qquad \text{and} \qquad \int_a^x f(t)dt?$$

Fundamental Theorem of Calculus, Part II. If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Illustrations.

$$\int_0^\pi \cos t dt \qquad \int_0^{\ln 2} e^{3x} dx \qquad \int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}} du$$