Calculus & Analytic Geometry I

Functions Form the Foundation

What is a function?

A **function** is a rule that assigns to each element x (*called the input or independent variable*) in a set D exactly one element $f(x)$ (*called the ouput or dependent variable*) in a set E.

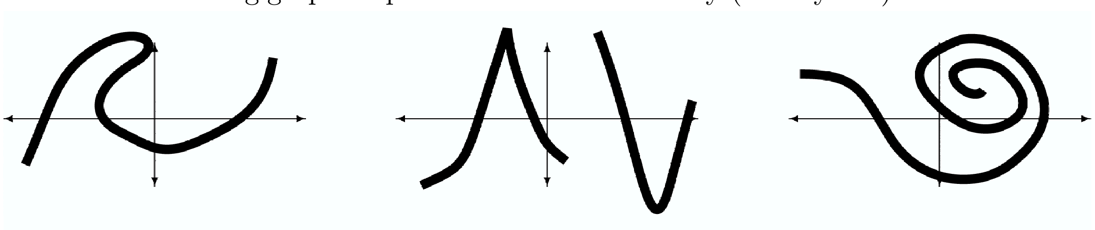
D: The set of possible values for the input variable is called the **domain.**

E: The set of all possible output values for $f(x)$ is called the **range.**

Note: The definition says *rule* and not *formula.* Why make this distinction?

Examples of functions:

Which of the following graphs represent functions and why (or why not)?



Linear Functions.

A function is **linear** if any change in the input causes a proportional change in the output.

So y is a linear function of x means that

any change in y	is proportional to	the corresponding change in x
Δy	$\propto$	Δx
Δy	$=$	$m\Delta x$

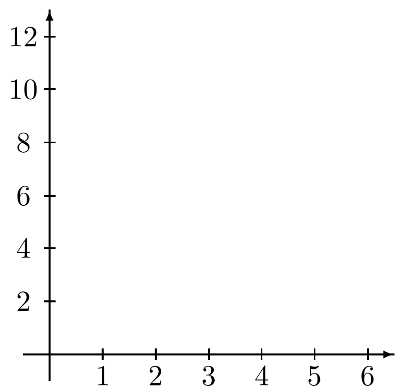
where m is the constant of proportionality.

Problem.

1. Suppose that y is a linear function of x. If $y = 2$ when $x = 0$ and $y = 5$ when $x = 2$, what is the constant of proportionality?

2. What is y when $x = 3$? when $x = 7$?

3. Plot the known values of y versus x on the graph below.



Formulas of a line.

The common formulas for a line can be derived from the equation of proportionality $\Delta y = m\Delta x$. Assume that (x_0, y_0) is a fixed point on the line, (x, y) is an arbitrary point, and m is the constant of proportionality. Then

$$\Delta y = m\Delta x$$

point slope form

initial value form

slope–intercept form

The constant m has many different interpretations.

$m \leftrightarrow$ slope
constant of proportionality
multiplier
rate of change of y with respect to x

Calculus & Analytic Geometry I

Essential Functions—Ghosts of Mathematics Past

- linear functions

- polynomial functions

- power functions

- rational functions

- trigonometric functions

- exponential functions

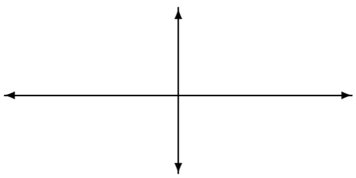
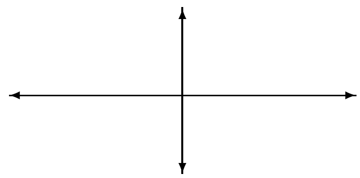
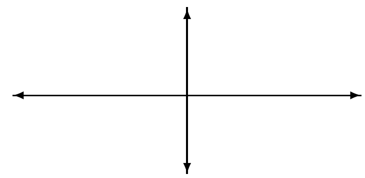
- logarithmic functions

Calculus & Analytic Geometry I

Making New Functions from Old Friends

—Shifting—

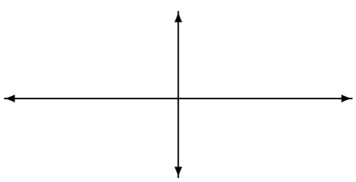
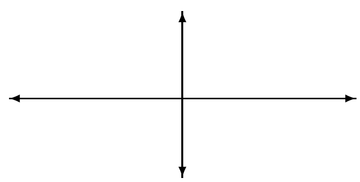
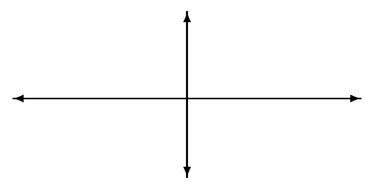
$f(x) = \ln(x)$ $\quad g(x) = \ln(x) + 2$ $\quad h(x) = \ln(x - 1)$

How would you change $f(x)$ to shift the graph down 3 units vertically?
How would you change $f(x)$ to shift its graph 2 units to the left?

—Stretching—

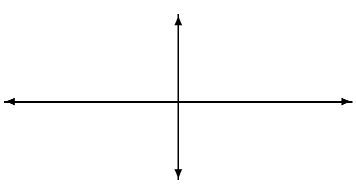
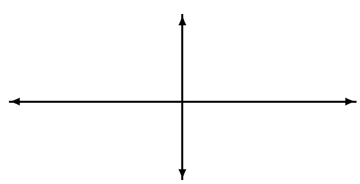
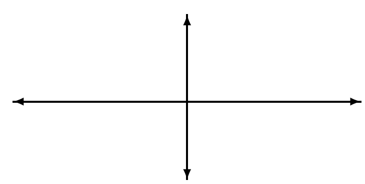
$f(x) = \cos(x)$ $\quad g(x) = 3\cos(x)$ $\quad h(x) = -2\cos(x)$

—Composition: Functions of functions—

The output of the function $f \circ g(x) = f(g(x))$ can be determined as by using the output of the function g as the input for the function f.

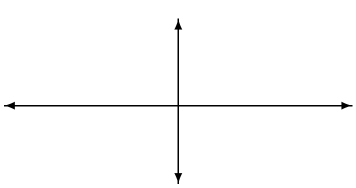
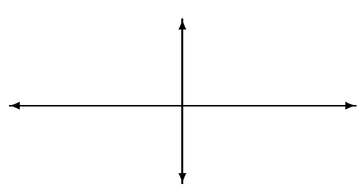
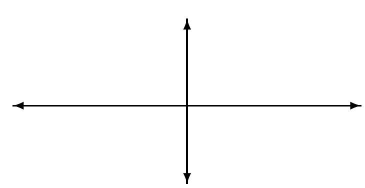
$f(x) = 2^x$ $\quad g(x) = x - 1$ $\quad h(x) = f \circ g(x)$

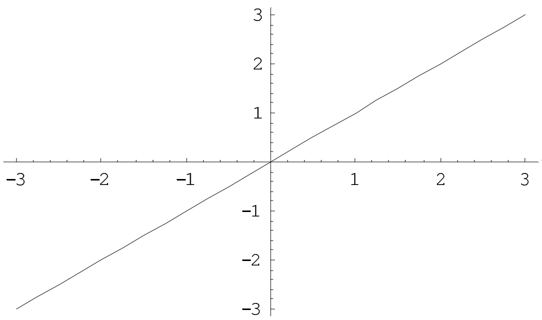
—Odd/ Even/ Neither—

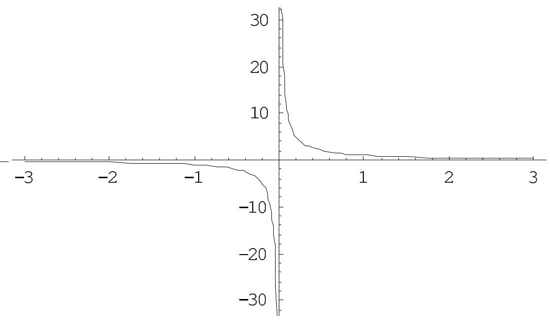
A function $f(x)$ is called *odd* provided $f(-x) = -f(x)$ for all values of x. A function $f(x)$ is called *even* provided $f(-x) = f(x)$ for all values of x.

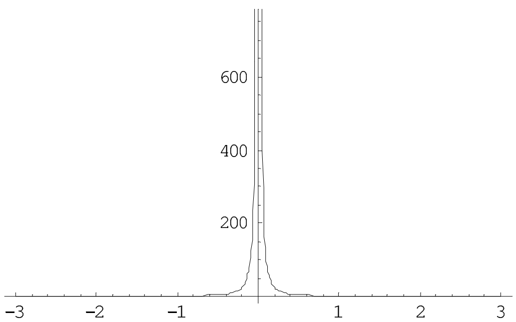
$f(x) = |x - 1|$ $\quad g(x) = \tan(x)$ $\quad h(x) = 2\cos(x)$

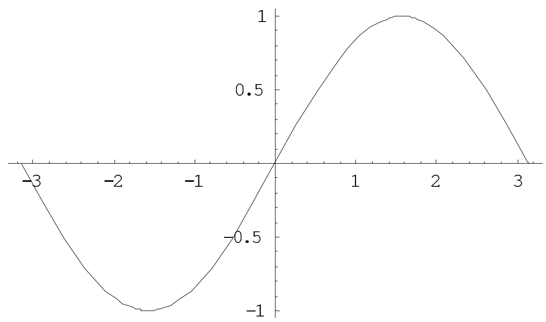




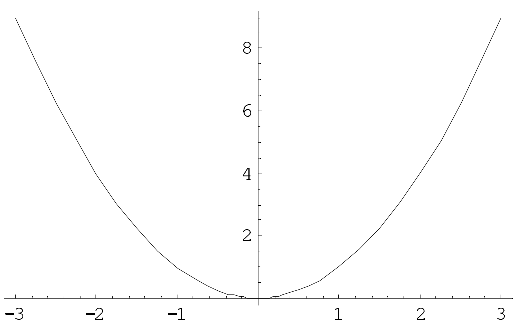
Make your best guess as to the function represented here. (e.g. Linear? Polynomial of degree n? Rational? Trigonometric? Exponential? Logarithmic?)

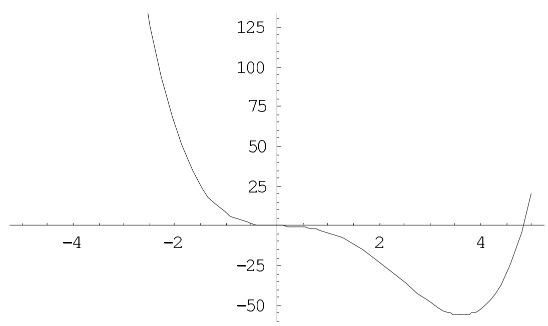


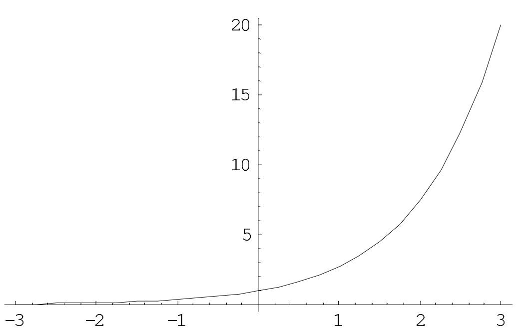


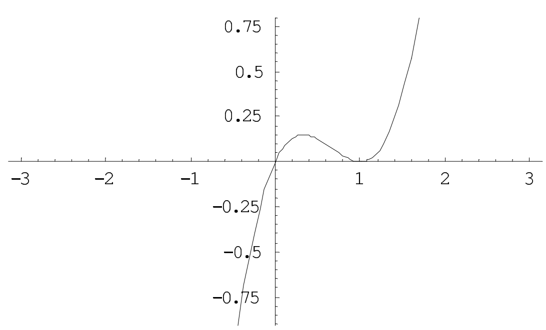












Calculus & Analytic Geometry I

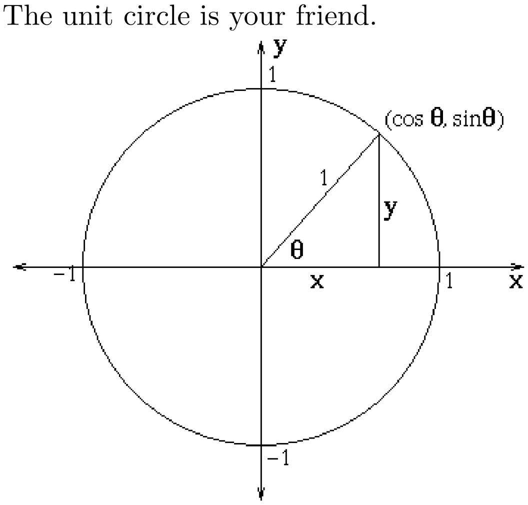
Brief Trig Encounter

In all likelyhood, we will not address this page in class. You are expected to be familiar with all trigonometric functions—especially the interpretation of the sine and cosine as the coordinates on a unit circle. I have provided this sheet for your benefit. You might want to work through it with friends (or come to office hours and discuss). Yes, you are expected to know this material.

—Shift,stretching review—
Sketch $f(x) = -2\sin(2x - \frac{\pi}{2})$ by successively graphing

$\sin(x)$
$\sin(2x)$
$\sin(2x - \frac{\pi}{2})$
$-2\sin(2x - \frac{\pi}{2})$.

Always work in radians! (radius units)
The unit circle is your friend.

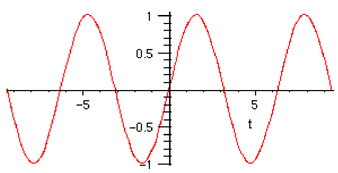


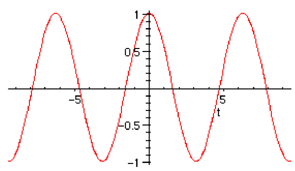
$\cos^2 x + \sin^2 x = 1$

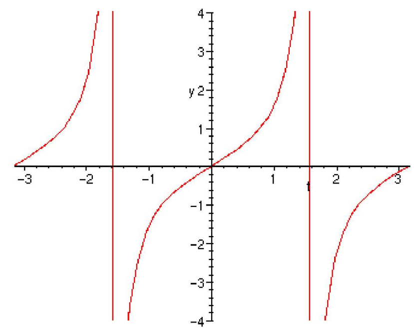
	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	$\sqrt{0}/2$	1	0
$\pi/6$	$\sqrt{1}/2$		
$\pi/4$	$\sqrt{2}/2$		
$\pi/3$	$\sqrt{3}/2$		
$\pi/2$	$\sqrt{4}/2$		
π	0	-1	0

Quick Check
$\sin(5\pi/6)$ $\tan(4\pi/3)$ $\cos(13\pi/4)$

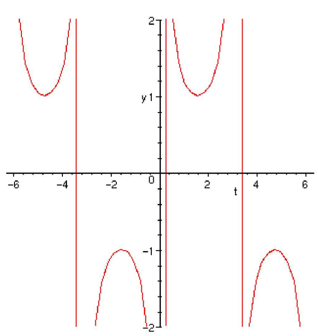
	$\sin(x)$	$\cos(x)$	$\tan(x)$
Domain :	x real $-\infty < x < \infty$	x real $-\infty < x < \infty$	x real $x \neq \pm\pi/2, \pm 3\pi/2, \ldots$
Range :	$-1 \leq x \leq 1$		
Period :			
Odd/Even?			
Sketch :			

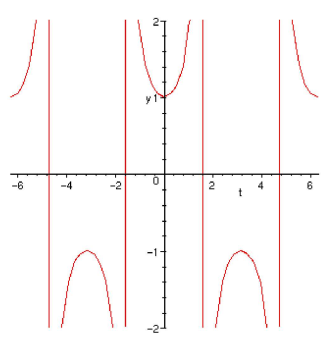


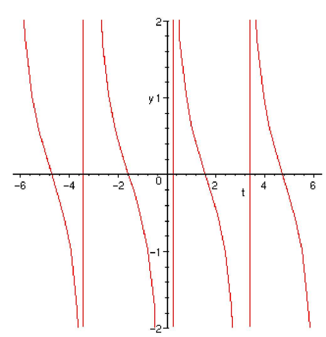




	$\csc(x)$	$\sec(x)$	$\cot(x)$
Domain :	x real	x real	x real
Range :			
Period :			
Odd/Even?			
Sketch :			







$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

Exponential Functions

Trouble with Tribbles. From Star Trek lore, we know that tribbles are fuzzy hermaphrodites about the size of an English muffin. Every month, a single tribble produces a litter of three offspring.

If we begin with one tribble at the beginning of the first month, how many tribbles are there at the end of the first month?

At the end of the second month?

At the end of the third month?

At the end of one year?

Estimate the time until the size of the tribble population is larger than the current human population of the United States; current human population of earth. (According to the U.S. Bureau of the Census (`http://www.census.gov/main/www/popclock.html`) on September 23, 2008 at 18:15 GMT, the total population of the United States is approximately 305,253,370 and the World, 6,725,512,766.)

As you can see, we need to be good at manipulation exponential functions. Do you remember the following rules of exponents?

Review of Algebra		
$a^0 = 1$	$a^{-1} = 1/a$	$a^{-x} = \dfrac{1}{a^x}$
$a^x \cdot a^t = a^{x+t}$	$\dfrac{a^x}{a^t} = a^{x-t}$	$(a^x)^t = a^{xt}$

Question. Compare and contrast exponential functions to power functions?

IF A FUNCTION IS SAID TO HAVE A DOUBLING TIME/TRIPLING TIME/HALF LIFE/ETC. THAT FUNCTION MUST BE EXPONENTIAL!

Question. Why do we make such a big deal about $e \approx 2.718281828...$? Can't we use any base (like 4 on the previous example)?

Sketch the shifted curves.

$y = e^x$ $\quad$ $y = e^{-x}$ $\quad$ $y = e^{x+1}$ $\quad$ $y = e^{-x-1}$

Simplify.

$9^{1/3} \cdot 9^{1/6}$ $\quad$ $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$ $\quad$ $\left(\frac{2}{\sqrt{2}}\right)^4$

Calculus & Analytic Geometry I

Inverse Functions and Logarithms

Unifying Idea An inverse of a function *undoes the action* of the function on it's input. So

$$\text{if } f(a) = b \text{ then } f^{-1}(b) = a.$$

Said another way, $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$.

Examples. $f(x) = x + 2$ $g(x) = 5x$ $h(x) = \sqrt{x}$

Not every function has a well-defined inverse.

$f(x) = \sin(x)$ $g(x) = x^2$ $h(x) = 2$

Important Property. A function is *one-to-one* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. In other words....

Are the following functions 1-1 on the given domains?

$\cos(x)$ on $[0, \pi]$ $\sin(x)$ on $[0, \pi]$ $\frac{1}{x}$ on all reals save $x = 0$.

If f is a 1-1 function on a domain B with range R, then f^{-1} exists and is a function from domain R to range B.

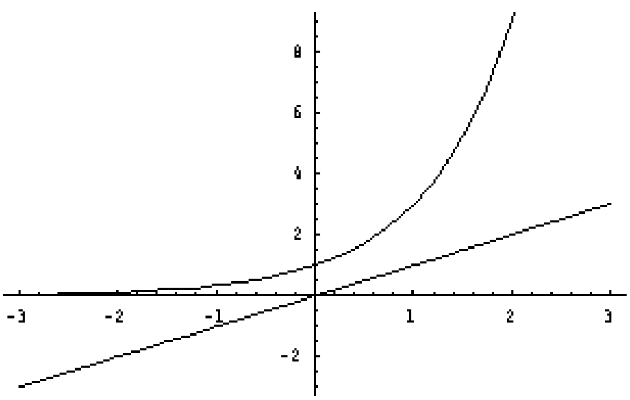
$$f : B \to R \qquad\qquad f^{-1} : R \to B$$

1. What is the domain of $f(x) = a^x$ (a a positive real number)?

2. Is $f(x) = a^x$ 1-1 on its domain? Why?

3. The inverse of f is defined to be $f^{-1} = \log_a(x)$.

$$y = a^x \qquad \Leftrightarrow \qquad \log_a(y) = x.$$



4. Simplify $5^{\log_5 7}$ $\qquad \log_e(e^x) \qquad \log_{121}(11) \qquad 2^{\log_4(x)}$.

Review of Algebra		
Using Exponents	Using Logarithms	General Information
$a^0 = 1$	$\log_c 1 = 0$	$\log_c x = y \leftrightarrow c^y = x$
$a^{-1} = 1/a$	$\log_c(AB) = \log_c(A) + \log_c(B)$	e is just a constant ≈ 2.71828
$a^{-x} = \dfrac{1}{a^x}$	$\log_c(A/B) = \log_c(A) - \log_c(B)$	$\log_e x = \ln x$
$a^x \cdot a^t = a^{x+t}$	$\log_c(A^p) = p \log_c(A)$	$c^t = (e^{\ln c})^t$
$\dfrac{a^x}{a^t} = a^{x-t}$	$\log_c(c^x) = x$	$\ln(e^x) = x$
$(a^x)^t = a^{xt}$	$c^{\log_c x} = x$	$e^{\ln x} = x$

Problem. The half life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

- Express the amount of substance remaining as a funtion of time.
- When will there be 1 gram left?

Calculus & Analytic Geometry I

Speed, Tangent Lines, and Successive Approximation

Average vs. Instantaneous Speed. According to MapQuest, it should take 3 hours and 42 minutes to travel the 226 mi between the University of Washington, Tacoma and Oregon State University. What would be your average speed on this trip? How does that differ from the information on your speedometer?

speed vs. velocity?

Estimating the rate of change.

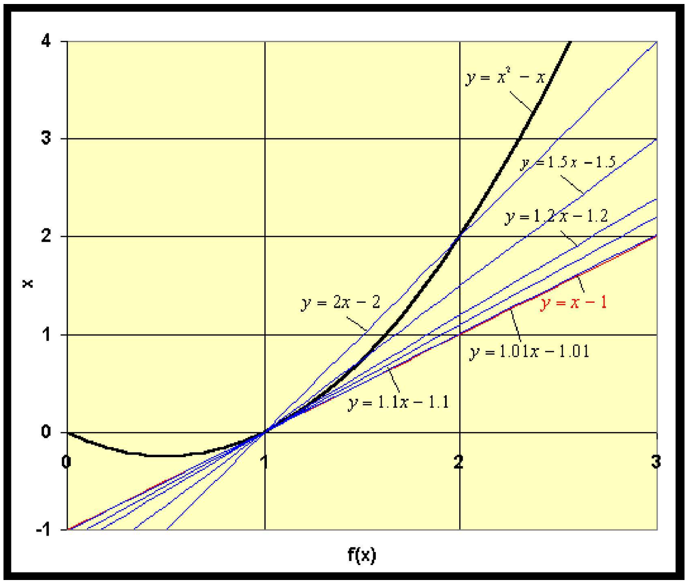
> The *average rate of change of a function* $y = f(x)$ *on an interval* $[a, b]$ is given by the change in the output divided by the change in the input:
>
> $$\frac{\Delta y}{\Delta x} = \frac{\text{change in output}}{\text{change in input}} = \frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h}$$
>
> if $h = b - a$.

What is the average rate of change of the function $f(x) = 3x + 2$ on the interval $[4, 10]$.

For a linear function, the rate of change (slope of the line) equals the average rate of change. For other functions, an average rate of change on an interval is used to *estimate* the rate of change of the function at a point inside that interval. The instantaneous rate of change gives the slope of the *tangent line* to the curve at a point.

Problem. Approximate the average rate of change for the function $f(x) = x^2 - x$ at the point $P(1, 0)$.

$[1, 2]$ $[1, 1.5]$ $[1, 1.1]$



As the interval of consideration shrinks, the distance between the points of intersection of the curve and the secant line ...

Definition. The *slope of a graph at a point* is the limit of the slopes seen in a microscope at that point, as the field of view shrinks to zero. (Also called the *rate of change of a function at a point.*)

Problem. Given the function

$$f(x) = \frac{2 + x^3 \cos(x) + 1.5^x}{x + x^2}$$

estimate the *rate of change* of the function at $x = 2$.

Given h, calculate $(f(2+h) - f(2))/h$.

Example. $h = 0.1$.

$$\frac{f(2.1) - f(2))}{0.1} = \frac{-.05104076048... - .153470884604...}{.1} \approx -2.0451...$$

So what is a LIMIT?

Calculus & Analytic Geometry I

Limits—Intuitively Thinking

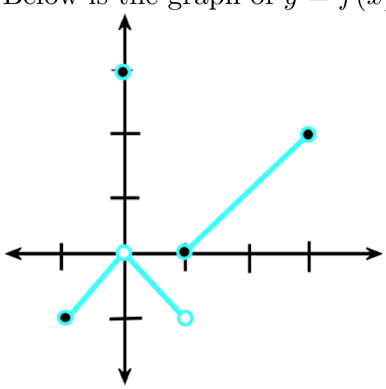
The *limit of a function* $f(x)$ *at a point* c is M, if the value of the function gets arbitrarily close to M for inputs close to c.

$$\lim_{x\to c} f(x) = M$$

Examples.

$\lim_{x\to 3} x^2 - 3$ $\qquad$ $\lim_{x\to 1} \frac{x^3-1}{x-1}$ $\qquad$ $\lim_{x\to 0} \frac{\sin x}{x}$

Below is the graph of $y = f(x)$.



Find the following limits if they exist:

$\lim_{x\to 0} f(x)$ $\qquad$ $\lim_{x\to 1} f(x)$

To have a limit at a point c, a function f must be defined on both sides of c. However we can consider the functions behavior from either side individually

$$\lim_{x\to c^+} f(x) = L \qquad \lim_{x\to c^-} f(x) = M$$

$\lim_{x\to -1^-} f(x)$ $\qquad$ $\lim_{x\to 0^-} f(x)$ $\qquad$ $\lim_{x\to 1^-} f(x)$ $\qquad$ $\lim_{x\to 3^-} f(x)$

$\lim_{x\to -1^+} f(x)$ $\qquad$ $\lim_{x\to 0^+} f(x)$ $\qquad$ $\lim_{x\to 1^+} f(x)$ $\qquad$ $\lim_{x\to 3^+} f(x)$

Theorem. A function $f(x)$ has a limit as x approaches c if and only if it has a left-handed and right-handed limit at c and these one-sided limits are equal.

Problem. Find $\lim_{z\to 3^+} \frac{\lfloor z \rfloor}{z}$ and $\lim_{z\to 3^-} \frac{\lfloor z \rfloor}{z}$

Vertical Asymptotes. Can a limit ever be infinite?

Let's see what happens as we approach the *holes* in the domain from the left and the right for

$$g(x) = \frac{e^x}{x^2} \qquad\qquad h(x) = \frac{2x^2 - x - 1}{3x^3 + 2x^2 - 5x}.$$

Writing $\lim_{x\to c^+} f(x) = \infty$ **does not mean** that ∞ is a number. It **does not mean** that the limit exists. It **means** that the *value of $f(x)$ becomes arbitrarily large as x approaches c from the right.*

If $\lim_{x\to c^+} f(x) = \infty$ and $\lim_{x\to c^-} f(x) = \infty$, we will go so far as to say

$$\lim_{x\to c} f(x) = \infty.$$

Similar meaning for $\lim_{x\to c} f(x) = -\infty$