Calculus & Analytic Geometry I

Limits Algebraically

Big Idea On *nice* functions, limits behave properly. (By that, I mean you can manipulate the symbol "lim" they way you would expect algebraically.)

The Laws. Suppose that c is a constant, n an integer, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

SUM $\lim\limits_{x\to a}[f(x)+g(x)] =$

DIFFERENCE $\lim\limits_{x\to a}[f(x)-g(x)] =$

CONSTANT MULTIPLE $\lim\limits_{x\to a}[cf(x)] =$

PRODUCT $\lim\limits_{x\to a}[f(x)g(x)] =$

QUOTIENT $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} =$

POWER $\lim\limits_{x\to a}[f(x)]^n =$

ROOT $\lim\limits_{x\to a}[f(x)]^{1/n} =$

$\lim\limits_{x\to a} c =$

$\lim\limits_{x\to a} x =$

Problem. Use the limit laws to evaluate:

$\lim\limits_{x\to -1} 3(x^2+1)^4$

$\lim\limits_{x\to 4^-} \sqrt{16-x^2}$

Other extremely useful properties.

- **Direct Substitution Property.** If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} =$$

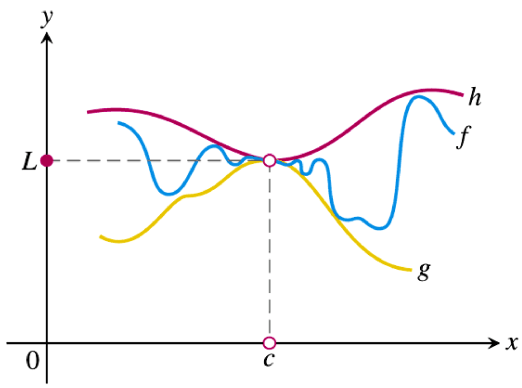
- If two functions are equal *except* at a fixed point a, their limits (if they exist) as they approach a must be equal.

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} \qquad \lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

- **The Sandwich Theorem.** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.



Example. Evaluate $\lim_{x \to 0} x^4 \cos \frac{2}{x}$

To think about. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to 0. What does this tell you about $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$?

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Continuity

Nice functions are continuous....

Definition. A function $f(x)$ is *continuous at a(n interior) point* c, if

$$\lim_{x\to c} f(x) = f(c).$$

A function is *continuous at a left endpoint* a or is *continuous at a right endpoint* b if

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x\to b^-} f(x) = f(b).$$

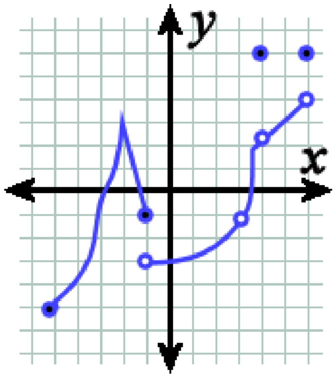
How can a function fail to be continuous at a point?

Definition. A function is called *continuous* if it is continuous at every point of its domain.

Is $f(x) = 1/x^2$ continuous?

Continuous functions include ... *Theorem 7, pg 124*

Where does the following function fail to be continuous on the interval $[-5, 6]$?



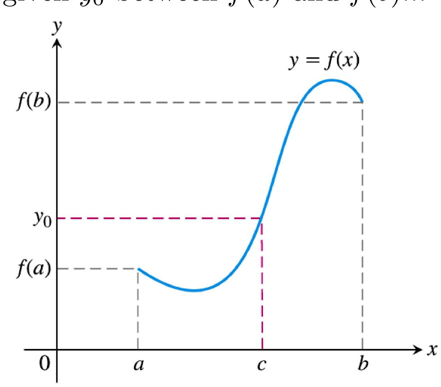
Good News—Algebraic combinations of continuous functions are continuous (whenever they are defined). So if f and g are continuous at $x = c$, then so are $f + g$, $f - g$, $f \cdot g$, $k \cdot f$ for a constant k, f/g provided $g(c) \neq 0$, and $f^{r/s}$ provided it is defined and r and s are integers. Inverses of continuous functions are continuous. Compositions of continuous functions are continuous.

Problems. For what real numbers do these functions fail to be continuous?

$$y = \frac{1}{x-2} - 3x \qquad\qquad y = \frac{1}{|x|+1} - \frac{x^2}{2}$$

$$y = \frac{\sin x}{x} \qquad\qquad y = \sqrt{2x+3}$$

Intermediate Value Theorem for Continuous Functions A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. Said another way, given y_0 between $f(a)$ and $f(b)$...



Problem. Prove that there is a root of the equation $x^4 + x = 3$ in the interval $(1, 2)$.

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Horizontal Asymptotes

Limits at positive or negative infinity tell us how the function "eventually" behaves (if ever)...

$f(x) = e^{-x}$ $\qquad g(t) = \dfrac{7t^3}{t^3 - 3t^2 + 6t}$ $\qquad h(z) = \dfrac{2 + \sqrt{z}}{2 - \sqrt{z}}$

$f(x) = \dfrac{2x^3 - 2x + 3}{3x^3 + 2x^2 - 5x}$ $\qquad h(x) = \dfrac{2x^2 - 2x + 3}{3x^3 + 2x^2 - 5x}$ $\qquad g(x) = \dfrac{2x^4 - 2x + 3}{3x^3 + 2x^2 - 5x}$

A line $y = b$ is a *horizontal asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

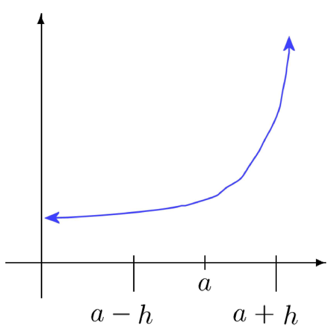
Problem. Find the asymptotes (horizontal and vertical) for $f(x) = \dfrac{x^3 + 7x^2 - x - 7}{x^2 + x - 2}$.

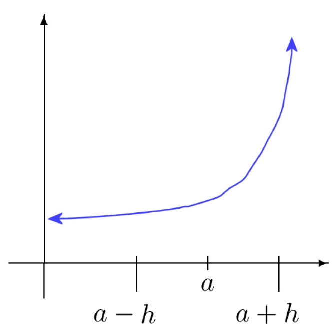
Tangents and Derivatives at a Point

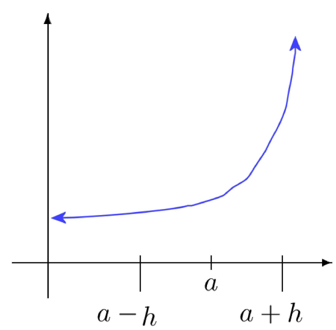
Informally, the *derivative* of a function $y = f(x)$ at a point $x = a$ is defined to be the slope (or the rate of change) of the function at the point $(a, f(a))$. In §2.1, we estimated the rate of change (a.k.a. the derivative) by calculating the average rates of change of the function on smaller and smaller intervals containing a and watched the rates stabilize.

For each of the intervals given below, write an equation representing the average rate of change for the general function $y = f(x)$ on the specified interval. Then, on the graph provided sketch the line used to calculate the average rate of change on that interval.

Interval:	$(a-h, a)$	$(a, a+h)$	$(a-h, a+h)$
Avg. rate of change:			







The expressions you have just discovered are referred to as difference quotients. *Each expression* acts as an estimate for the derivative $f'(a)$. To improve these estimates, let $h \to 0$. This leads us to the formal definition of the derivative.

Definition. The *derivative (slope, rate of change)* of a function f at $x = a$ is defined as:

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} \end{aligned}$$

provided these limits exist and are equal. In this case, f is said to be *differentiable* at $x = a$.

Are all three difference quotients really needed? Consider $f(x) = |x|$ at $x = 0$.

Problems. Find the slope of the function's graph at the given point. Then find the equation for the line tangent to the graph there.

1. $y = 2\sqrt{x}, \qquad (1, 2)$

2. $y = \dfrac{x-1}{x+1}, \qquad x = 0$

Summary. The following ideas are equivalent:

- the slope of $y = f(x)$ at $x = x_0$
- the slope of the tangent to the curve $y = f(x)$ at $x = x_0$
- the rate of change of $f(x)$ with respect to x at $x = x_0$
- the derivative $f'(x_0)$
- the limit of (any) difference quotient, $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

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The Derivative as a Function

Problem. Find the slope of the function's graph at the given point. Then find the equation for the line tangent to the graph there.

1. $y = \dfrac{x-1}{x+1}, \qquad x = 0$

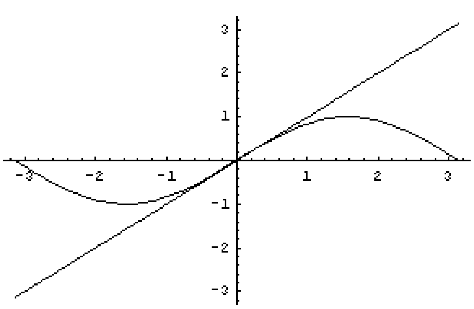
Working Backwards. What derivative is represented by each of the following expressions?

$$\lim_{x\to 5}\frac{2^x - 32}{x-5} \qquad\qquad \lim_{h\to 0}\frac{\cos(\pi + h) + 1}{h}$$

Derivative as a function. Up to now, we have looked at the derivative as a *number*. It gives us information about a function at a *point.* But the numerical value of derivative varies from point to point, and these new values can also be considered as the values of a new function— the derivative function—with its own graph. Viewed in this way the derivative is a global object.

All functions and their derivatives are related in the same way.

Function	↔	Derivative
increasing	↔	
decreasing	↔	
horizontal	↔	
steep (rising or falling)	↔	
gradual (rising or falling)	↔	
straight	↔	



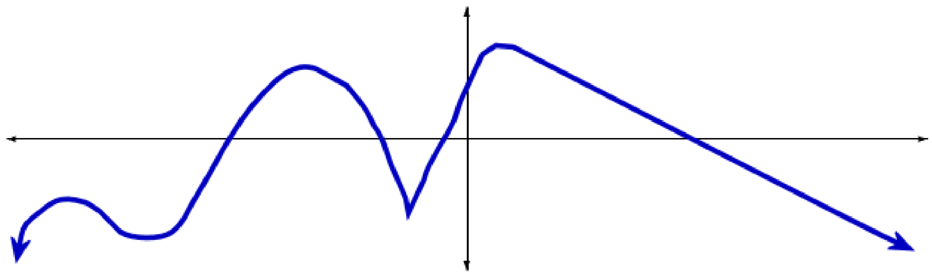
Notation. Many ways to denote the derivative of the function $y = f(x)$:

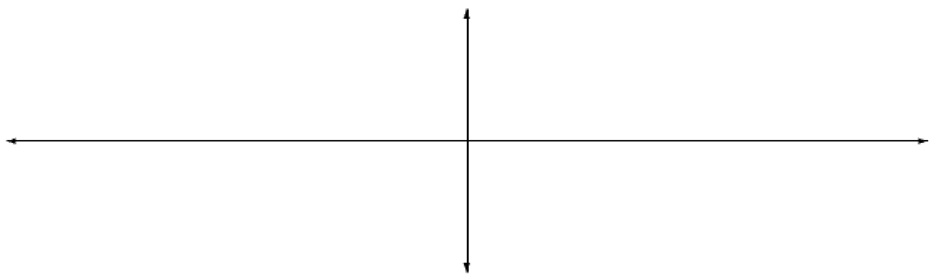
$$f'(y) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

To indicate the value of the derivative at a specified point $x = a$:

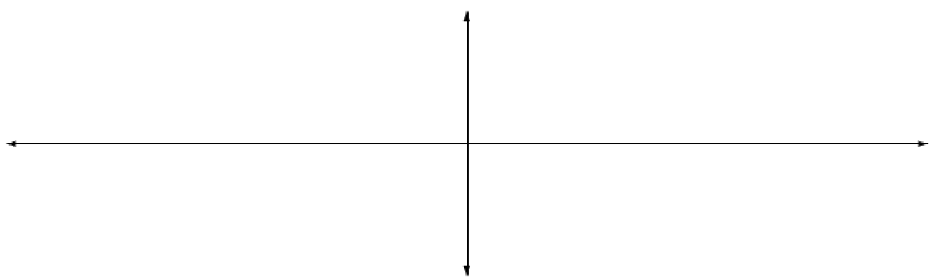
$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Illustration. For the graph below, sketch its derivative.





Now make a sketch of a graph which would have the original graph as its derivative.



As time permits: For each of the elementary functions below, first make a rough sketch of the function itself, then, based on the correspondences between functions and their derivatives, sketch the derivative of the function (as best you can.)

1. $f(x) = 3x$

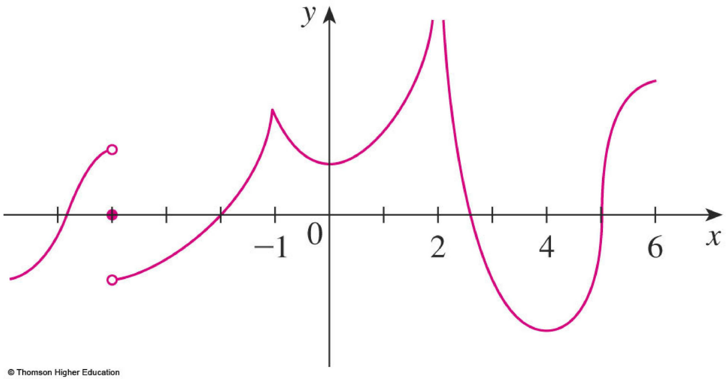
2. $g(x) = 3x + 1$

3. $h(x) = x^2$

4. $i(x) = e^x$

5. $j(x) = \sin x$

Definition. A function f is *differentiable at a* is $f'(a)$ exists. It is *differentiable on an open interval* (a, b) if it is differentiable at every number in the interval. A function is *differentiable* if it is differentiable at every point in its domain.



For your consideration

[T/F] If f is differentiable at a, then f is continuous at a.

[T/F] If f is continuous at a, the f is differentiable at a.

Higher Derivatives Since the derivative of a function f gives a new function f', there is nothing stopping us from analyzing the rate of change of f', denoted f'' or d^2f/dy^2. What about the rate of change of f''?