# Calculus & Analytic Geometry I

## The Derivative: Analytic Viewpoint

**Derivative of a Constant Function.** For $c$ a constant, the derivative of $f(x) = c$ equals

$f'(x) =$

**Derivative of a Linear Function.** If $f(x) = mx+b$, then $f'(x) =$ ____________________.

**Derivative of a Constant Times a Function.** If $f(x) = c \cdot g(x)$, then $f'(x) =$ ____________________.

**Proof.** $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

**Derivatives of Sums and Differences.** If $f(x) = g(x)+h(x)$, then $f'(x) =$ ____________________.

**Proof.** $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

So if $f(x) = g(x) - h(x)$, then $f'(x) =$ ________________ because

**Derivative of a Power Function.** To calculate the derivative of $x^2$ (using the righthanded difference quotient) we had to multiply out $(x+h)^2$. In general, to find the derivative of $x^n$ (for $n$ and integer) we will have to multiply out $(x+h)^n$. Let's look at some examples:

$$\begin{aligned}
(x+h)^2 &= x^2 + 2xh + (h)^2 \\
(x+h)^3 &= x^3 + 3x^2h + 3x(h)^2 + (h)^3 \\
(x+h)^4 &= x^4 + 4x^3h + 6x^2(h)^2 + 4x(h)^3 + (h)^4 \\
\vdots \quad &\quad \vdots \\
(x+h)^n &= x^n + nx^{n-1}h + \underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + (h)^n}_{\text{Terms involving } (h)^2 \text{ and higher powers of } h}
\end{aligned}$$

Now to find the derivative of $f(x) = x^n$:

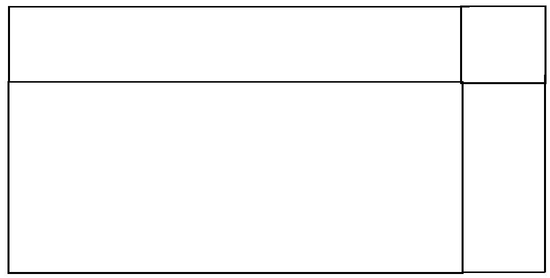
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

**Derivative of an Exponential.** If $f(x) = e^x$, then $f'(x) =$ ____________________.

**Proof.** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

**Derivation of Product Rule.** Suppose we know the derivatives of $f(x)$ and $g(x)$ and we want to calculate the derivative of the product $f(x)g(x)$.

$$(f(x)g(x))' = \lim_{h \to 0}$$



**The Product Rule.**

$$(fg)' = f'g + fg'$$

In words:

> The derivative of a product is the derivative of the first factor multiplied by the second, plus the first factor multiplied by the derivative of the second.

**Quotient Rule**. For completeness...Suppose we know the derivatives of $f(x)$ and $g(x)$ and we want to calculate the derivative of the quotient $f(x)/g(x)$.

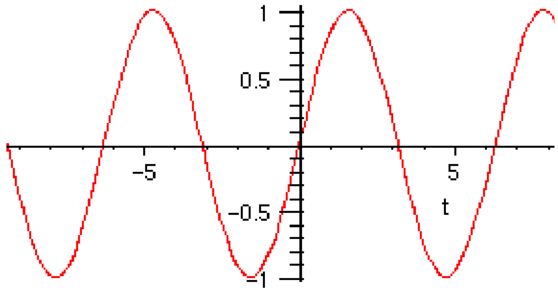
$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

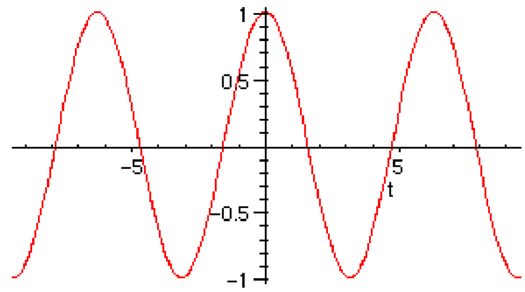
We could derive it with difference quotients, but it will be much easier when we have the chain rule....

**Apply your knowledge and find the following derivatives:**

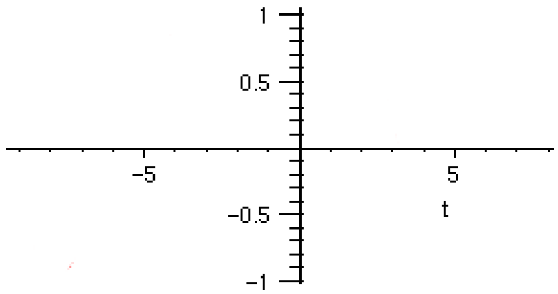
1. $y = (3 - x^2)(x^3 - x + 1)$
2. $v = \dfrac{1 + w - 4\sqrt{w}}{w}$
3. $z = \dfrac{12}{x^2}$

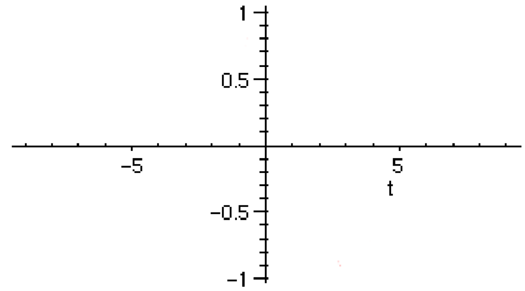
Recall the graphs of your favorite trig functions...




Now sketch their derivative functions based one where they are increasing, decreasing, or have a slope of zero.




So we conjecture that

$$\frac{d}{dx} \sin x =$$

$$\frac{d}{dx} \cos x =$$

Apply this knowledge.

1. Find $\frac{d}{d\theta}(\theta \sin\theta + \cos\theta)$.

2. Find $[(\sin t + \cos t)(\sin t - \cos t)]'$.

3. Find the derivatives for the remaining trigonometric functions.

$(\tan x)' =$

$(\sec x)' =$

$(\cot x)' =$

$(\csc x)' =$

Last but not least, it is time to analytically verify that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$. Recall the sum formulas:

$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$ $\qquad$ $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h}$$

$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x-h)}{2h}$$

# Calculus & Analytic Geometry I

## The Chain Rule—Derivative of function compositions

**The Chain Rule** — Motivation.

Suppose we are blowing up a spherical balloon. We know that the volume of a balloon depends its radius. ($V = \frac{4}{3}\pi r^3$.) When the radius is 15 cm, at what rate is the volume changing with respect to a change in the radius?

As we blow up the balloon, the radius (and hence the volume) are changing over time. Suppose the radius is changing at a rate of 3 cm every second when $r = 15$ cm. When the radius is 15 cm, at what rate is the volume changing with respect to a change in time?

**The Chain Rule** — Derivation.

The chain rule applies to a composition of functions. Suppose $f(g(x))$ is a composite function. Let us write

$$z = g(x) \text{ and } y = f(z), \text{ so } y = f(g(x)).$$

How does a change in $x$ *approximately* effect a change in $z$?

How does a change in $z$ *approximately* effect a change in $y$?

Combine these two approximations to relate a change in $y$ to a change in $x$.

**In words:** The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

**Problems.** Find the derivatives of the given functions:

1. $f(x) = \sqrt{1 - x^2}$

2. $z = 3^{-6t}$

3. $h(r) = \sin(10r + 3)$

4. $p(x) = \sec x$

5. $\ell(\theta) = \sin 5\theta + \cos^2 \theta$

6. $g(s) = \sec^3(4s)e^{\sin s}$

7. The quotient rule as an application of the product rule and the chain rule.
   $P(x) = f(x) \cdot g^{-1}(x)$

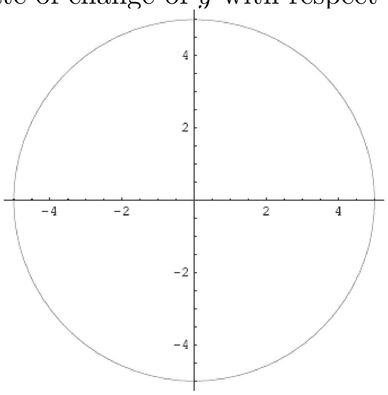
# Calculus & Analytic Geometry I

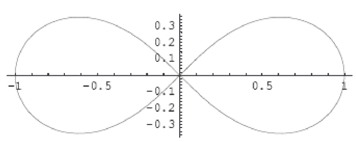
## Implicit Differentiation

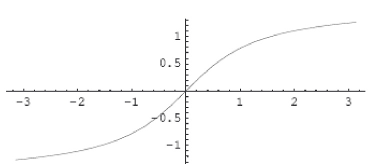
Sometimes we are faced with equations that *imply* a relation between two variables.

$$x^2 + y^2 = 25 \qquad (x^2 + y^2)^2 = x^2 - y^2 \qquad x = \tan(y)$$

Sometimes we can solve for $y$ in terms of $x$ and sometimes we can't. But we can still ask about the rate of change of $y$ with respect to $x$. We treat $y$ as a function of $x$ and use the chain rule.



## Derivatives of Inverse Functions

The chain rule is a powerful differentiation tool. It helps us determine slopes of

- composition of functions
  Find $\frac{d}{dx}\sec^2(x)$.

- parametric functions
  Suppose $x(t) = 2t^2 + 3$ and $y(t) = t^4$. Find $\frac{dy}{dx}$ at $t = -1$.

- implicitly defined functions
  Find $\frac{dr}{d\theta}$ if $e^{r^2\theta} = 2r + 2\theta$.

- inverse functions—as we shall see today.

Find the derivative of $\ln(x)$ by differentiating the identity
$e^{\ln(x)} = x$.

**Problem.** Find derivatives for the following functions:
$y = \ln(\sec x)$ $\qquad$ $y = \ln[t(t+1)(t+2)(t+3)]$

**Logarithmic differentiation.** When you need to find a derivative involving lots of products or quotients, sometimes taking a logarithm first can help.

Find derivative of $y = \dfrac{\theta \sin\theta}{\sqrt{\sec\theta}}$.

Find the derivative of $f^{-1}(x)$ in terms of $f'(x)$ by differentiating the identity $f(f^{-1}(x)) = x$.

**Check yourself.** Assume that $f(x)$ and $g(x)$ are inverse functions and

$$\begin{array}{lcrlcr} f(-2) & = & 1 & \qquad f'(-2) & = & 3 \\ f(1) & = & 7 & \qquad f'(1) & = & -10 \\ f(7) & = & -2 & \qquad f'(7) & = & -2 \end{array}$$

What is $g(7)$? Find $g'(-2)$.

**Final Note.** For $a > 0$ and $u$ a differentiable function of $x$,
$\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$ and $\dfrac{d}{dx}\log_a u = \dfrac{1}{u \ln a}\dfrac{du}{dx}$.