Calculus & Analytic Geometry I

Applications of Derivatives to Other Disciplines

Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them. —Joseph Fourier (1768-1830)

velocity	rate of change of displacement with respect to time
acceleration	rate of change of velocity with respect to time
linear density	rate of change of mass with respect to length
current	rate at which charge flows through a surface
rate of reaction	rate of change in concentration over time
compressibility	rate of change of volume with respect to pressure per unit volume
growth rate	rate of change of population over time
marginal cost	rate of change of cost with respect to the number of items produced
marginal revenue	rate of change of revenue with respect to the number of items produced
marginal profit	rate of change of profit with respect to number of items produced
sensitivity	rate of change of reaction with respect to the strength of some stimulus

1. **Profit Analysis.** A national toy distributor determines the cost and revenue models for one of its games.

$$C = 2.4x + .002x^2, \quad \text{for} \quad 0 \le x \le 6000$$
$$R = 7.2x - .001x^2, \quad \text{for} \quad 0 \le x \le 6000.$$

Determine the interval on which the profit function is increasing. Interpret the marginal profit when $x = 2000$.

2. **Lunar projectile motion.** A rock is thrown vertically upward from the surface of the moon at a velocity of 24 m/sec reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

   (a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)

   (b) How long does it take the rock to reach its highest point?

   (c) How high does the rock go?

   (d) How long does it take the rock to reach half its maximum height?

   (e) How long is the rock aloft?

3. **Learning Curve.** The rate at which a postal clerk can sort mail is a function of the clerk's experience. Suppose the postmaster of a large city estimates that after t months on the job, the average clerk can sort $Q(t) = 700 - 400e^{-.5t}$ letters per hour.

   (a) How many letters can a new employee sort per hour?

   (b) How many letters will a clerk with 6 months experience sort per hour?

   (c) Approximately how many letter will the average clerk ultimately be able to sort per hour?

Calculus & Analytic Geometry I

Exponential Growth and Decay

When quantities grow (or decay) at a rate proportional to their size, we have

$$f'(t) = k \cdot f(t)$$

for some constant k that represents the *relative growth rate.* The only solution to this type of differential equation has the form

$$f(t) = Ce^{kt}.$$

Verify that $f(t) = Ce^{kt}$ **is a solution to** $f'(t) = k \cdot f(t)$.

1. **Growth.** A bacterial culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours their count is 75,000.
   (a) Find the initial population.
   (b) Find an expression for the population after t hours.
   (c) Find the number of cells after 5 hours.
   (d) Find the rate of growth after 5 hours.
   (e) When will the population reach 200,000?
2. **Decay.** A sample of tritium-3 decayed to 94.5% of its original amount after a year.
   (a) What is the half-life of tritium-3?
   (b) How long would it take the sample to decay to 20% of its original amount?
3. **Newton's Law of Heating/Cooling**. The rate of heating/cooling of an object is proportional to the temperature difference between the object and its surroundings.
   $$\frac{dT}{dt} = k(T - T_s).$$
   Let $y = T - T_s$.

   A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°. If the temperature of the turkey is 150°F after 30 minutes, what is the temperature after 45 minutes? When will the turkey have cooled to 100°F?
4. **Interest.** If $ 1000 is borrowed at 6.25% interest, find the amounts due at the end of 3 years if the interest is compounded (a) annually, (b) quarterly, (c) monthly, (d) weekly, (e) daily, (f) hourly, (g) continuously.

Calculus & Analytic Geometry I

Related Rates

Related Rates are word problems based on the chain rule (much like our motivating balloon problem.) The goal is to find a rate of change from other known rates.

Strategy.

- Draw a picture and label the variables and constants.
- Write down the given information.
- Write down what you are asked to find.
- Write an equation that relates the variables.
- Differentiate with respect to t.
- Evaluate using known values to find unknown rate.

Problems

1. Suppose that the edge length x, y, and z of a closed rectangular box are changing at the following rates:
   $\frac{dx}{dt} = 1$ m/s $\frac{dy}{dt} = -2$ m/s $\frac{dz}{dt} = 1$ m/s.
   Find the rates at which the box's diagonal length is changing when $x = 4$, $y = 8$, and $z = 2$.

We could have easily asked to find the rates at which the box's area or surface area are changing when $x = 4$, $y = 3$, and $z = 2$. Could you now solve these problems?

2. A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 3 ft/sec. How fast is the top of the ladder sliding down the wall then?

   At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

   At what rate is the angle θ between the ladder and the ground changing then?

3. A man 6 ft tall walks at a rate of 5 ft/sec toward a street light that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the light?

Calculus & Analytic Geometry I

Linearization (A.K.A. Linear Approximation)

Linear Approximations. If a function f is "nice" at a point a, we can approximate the function near a by the equation of the line through $(a, f(a))$ with slope $f'(a)$. In other words, we use the equation of the tangent line to approximate the function near a.

Problem. Make a rough sketch of the graph $f(x) = \sqrt{x}$. Find the linearization of $f(x)$ at the point $(4, 2)$.

Estimate $\sqrt{4.0036}$.

How close to the true value of $\sqrt{4.0036}$ is our estimate using the linear approximation?

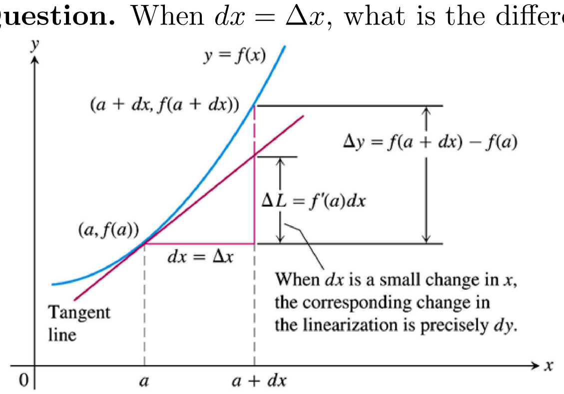
Differentials. Let $y = f(x)$ be a differentiable function. The **differential** dx is an *independent variable.* The **differential** dy is defined as

$$dy = f'(x)dx.$$

Problem

1. Find dy if $f(x) = \sqrt{x}$.
2. Find the value of dy when $x = 4$ and $dx = .0036$.

Question. When $dx = \Delta x$, what is the difference between Δy and dy?



So the linearization uses dy to approximate the true change Δy. Said another way...

$$f(a+\Delta x) = f(a) + \Delta y \approx f(a) + df.$$

Problems. Find dy for $y = (1+x)^k$ where k is a constant. Then estimate $(1.002)^{50}$ and $\sqrt[3]{1.009}$.

We can use differentials to analyze how error propagates through our calculations. Suppose we are calculating the area of a square by measuring the length of one of its sides and then using the formula $A = s^2$. Find dA when $s = 10$ cm.

If our measurement of the side is accurate to within 1 mm, how accurate is our calculation for the area?

The dA calculated above is the absolute error.
The *relative error* compares the absolute error to the calculated value.

The relative error in the measurement is $\dfrac{\Delta s}{s} = \dfrac{ds}{s} =$

The relative error in the area calculation is $\dfrac{dA}{A} =$

Local Approximation by Polynomials. For a function $f(x)$ the linear approximation of f near $x = a$ is the line

$$y = f(a) + f'(a)(x-a).$$

Find $y(a)$ and $y'(a)$.

A better approximation of $f(x)$ is a polynomial of higher order which exactly matches the values of the function and some more of its derivatives at the point $x = a$. These are called *Taylor polynomials.* The **$\mathbf{n}^{th}$-degree Taylor polynomial of f centered at a** is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Calculus & Analytic Geometry I

The Extremes

We are beginning Chapter 4: Applications of Differentiation. Today we examine the maximum and minimum values of functions (a.k.a. the extremes).

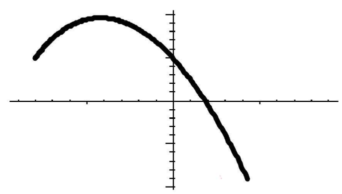
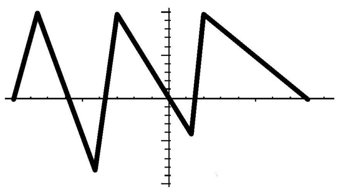
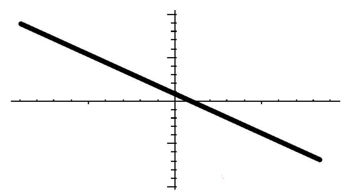
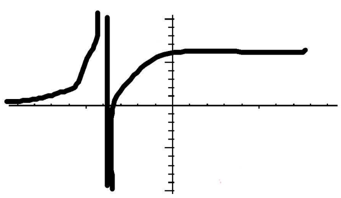
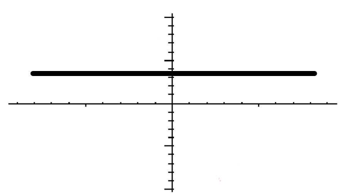
Definition. A function f with Domain D has an *absolute (global) maximum* at c if

$$f(x) \leq f(c) \text{ for all } x \text{ in } D$$

and an *absolute (global) minimum* at c if

$$f(x) \geq f(c) \text{ for all } x \text{ in } D.$$

The extreme value theorem says that a continuous function on a closed interval always has both an absolute maximum M and an absolute minimum m.

Definition. A function f with Domain D has a *local maximum* at c if

$$f(x) \leq f(c) \text{ for all } x \text{ in some open interval containing } c$$

and an *local minimum* at c if

$$f(x) \geq f(c) \text{ for all } x \text{ in an open interval containing } c.$$

What characteristics do extremes have?

Definition. A *critical point* is a point in the of the domain of a function f where either

- f' is zero, or
- f' is undefined.

Critical points are candidates for local extremes. Critical points and endpoints are candidates for global extremes.

Problems.

1. Find the absolute maximum and minimum value for $g(x) = xe^{-x}$ on the interval $-1 \leq x \leq 1$.

2. Find the absolute maximum and minimum value for $h(t) = 2 - |t|$ on the interval $-1 \leq t \leq 3$.

Calculus & Analytic Geometry I

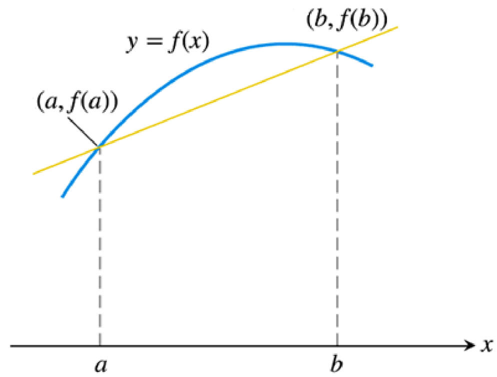
The Theorems—Most Especially the MVT

The Intermediate Value Theorem. A function $y = f(x)$ that is continuous on $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Rolle's Theorem. If a function $y = f(x)$ is

- continuous on $[a, b]$
- differentiable on (a, b)
- $f(a) = f(b)$

then there exists a c in (a, b) such that $f'(c) = 0$.



The Mean Value Theorem. If a function $y = f(x)$ is

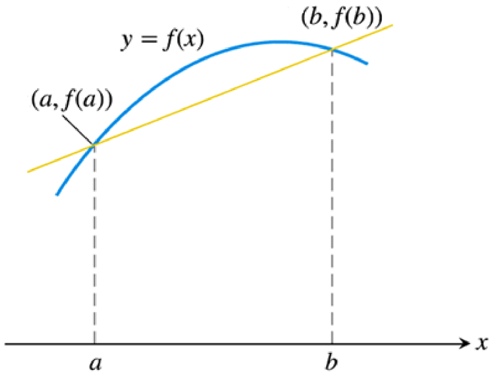
- continuous on $[a, b]$
- differentiable on (a, b)

then there exists a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof of Rolle's Theorem.

Application of Rolle's Theorem. Show that the function $f(x) = \frac{1}{1-t} + \sqrt{1+t} - 3.1$ has exactly one real zero in the interval $(-1, 1)$.

Proof of the MVT.



Let $h(x) = f(x) -$ (equation of secant line).

Corollary 1. If $f'(x) = 0$ at each point of (a, b), then $f(x)$ is a constant function on the interval (a, b). (i.e. $f(x) = C$ for some real number C.)

Corollary 2. If $f'(x) = g'(x)$ at each point of (a, b), then $f(x)$ and $g(x)$ differ by a constant function on the interval (a, b). (i.e. $f(x) = g(x) + C$ for some real number C.)

Problems.

1. Find all possible functions $f(x)$ such that $f'(x) = x^3$.
2. Find the function $g(t)$ such that $g'(t) = e^{2t}$ and $g(0) = \frac{3}{2}$.
3. Suppose the acceleration of a body is measure as 9.8 m/s^2 and we know that $v(0) = -3$ m/s, $s(0) = 5$ m. Find an equation to describe the body's position at time t.

The Derivative Tests

Question. How do you know where a function is increasing or decreasing?

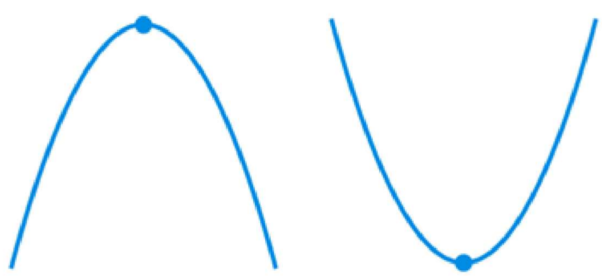
Question. How can a function behave at a transition point as it changes from increasing to decreasing or vice versa?

Question. How does this help us identity local extrema?

Consider a function whose derivative is $f'(x) = x^{-1/2}(x-3)$. Where is the function increasing? decreasing? Identify inputs that give local extremes.

Knowing how quickly the derivative of a function is changing can also give important information about local extremes.

Definition. The graph of a differentiable function $y = f(x)$ is *concave up* on an open interval I if f' is increasing on I and *concave down* if f' is decreasing on I.



Definition. A point where the graph of a function has a tangent line and where the concavity changes is called a *point of inflection*.

First Derivative Test. Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly c itself. Moving across c from left to right

- if f' changes from negative to positive at c, the f has a local ______________ at c;
- if f' changes from positive to negative at c, the f has a local ______________ at c;
- if f' does not change sign at c, then f has no local extremum at c.

Second Derivative Test. Suppose f'' is continuous on an open interval that contains $x = v$.

- If $f'(c) = 0$ and $f'' < 0$, then f has a local ______________ at $x = c$;
- If $f'(c) = 0$ and $f'' > 0$, then f has a local ______________ at $x = c$;
- If $f'(c) = 0$ and $f'' = 0$, then the test fails. The function f may have a local maximum, a local minimum or neither.

The rest of today and tomorrow, we will focus on sketching curves using these two tests.

Problem. Sketch the general shape of a curve satisfying the given information

interval:	$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
sign of f' :	$-$	$-$	$-$	$+$
sign of f'' :	$+$	$-$	$+$	$+$

Problem. Find all local extrema of $f(x) = -2\cos x - \cos^2 x$ on the interval $-\pi \le x \le \pi$.

Calculus & Analytic Geometry I

Indeterminant Forms and l'Hôpital's Rule

Application of derivatives to assess pesky limits...

Indeterminant Forms. Sometimes we need to evaluate $\lim_{x \to c} \frac{f(x)}{g(x)}$ where $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are either both 0 or both ∞ ($\frac{0}{0}$ or $\frac{\infty}{\infty}$).

Example. $\lim_{x \to 2} \frac{x^7 - 128}{x^3 - 8}$

l'Hôpital's Rule. Suppose that $f(c) = g(c) = 0$ and that f and g are differentiable on an open interval I containing c, and that $g'(x) \neq 0$ on I if $x \neq c$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Problems. Verify the following limits:

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}$
3. $\lim_{x \to 0} \frac{1 - \cos x}{\sec x} = 0$
4. $\lim_{x \to \infty} \frac{2x^2 - 3x + 1}{3x^2 + 5x - 2} = \frac{2}{3}$
5. $\lim_{x \to \infty} \frac{x + \sin x}{x - \cos x} = 1$
6. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$
7. $\lim_{x \to 0+} x^{\sin x} = 1$

The Indeterminant Forms: $\frac{0}{0}$ $\quad \frac{\infty}{\infty}$ $\quad 0 \cdot \infty$ $\quad \infty - \infty$ $\quad 0^0$ $\quad \infty^0$ $\quad 1^\infty$

Calculus & Analytic Geometry I

Curve Sketching

Problem. Sketch the general shape of a curve satisfying the given information

interval:	$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
sign of f' :	$-$	$-$	$-$	$+$
sign of f'' :	$+$	$-$	$+$	$+$

Strategies for Graphing Functions

- Identify domain and any symmetries the curve may have.
- Find first and second derivatives.
- Find critical points and identify behavior at each.
- Determine where function is increasing or decreasing.
- Find points of inflection and concavity.
- Identify asymptotes (l'Hôpital may come in handy).
- Plot key points (intercepts and anything found above).

More Problems. Graph as many of the following functions as time will permit.

1. $y = 4x^3 - x^4$
2. $y = 2x - 3x^{2/3}$
3. $y = e^{2/x}$
4. $y = \dfrac{(x+1)^2}{1+x^2}$

Calculus & Analytic Geometry I

Applied Optimization

Once again, word problems rear their ugly head—this time in the form of optimization problems. The strategies for solving word problems and global extremes both apply.

Strategies

- Identify to function to be optimized (i.e. find a absolute maximum or minimum) and its natural domain.
- Express the target function in terms of a single variable.
- Find critical points.
- Determine function value at critical points and end points of the domain. (Sometimes the second derivative test is useful—say if you know that the second derivative is always positive then you know that your critical point is a minimum.)
- Make sure that you answer the question asked. Write your answer as a complete sentence.

Applying the strategy.

1. A 50 cm wire is to be cut in two. The first piece will be used to form a circle and the second piece, a square. Where should the wire be cut so that the area of the resulting figures is maximized? minimized?
2. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light.
3. A rectangular plot of framed will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose and what are its dimensions?
4. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?
5. A corgi named Elvis waits on the shore of Lake Michigan for his owner, Tim Pennings, to throw a ball. The ball is thrown 10 m and lands in the water 6 m from the shore. If Elvis can run 6.4 m/s and swim 0.9m/s, what path should he follow to get to the ball the most quickly? Do dogs really know calculus?