Calculus & Analytic Geometry I

Antiderivatives and Initial Value Problems

Definition. A function F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$.

Question. Given a nice function f, how many antiderivatives can it have?

Definition The set of all antiderivatives of f is the *indefinite integral* of f with respect to x, denoted

$$\int f(x)dx.$$

The symbol $\int$ is an *integral sign.* The function f is the *integrand* and x is the *variable of integration.*

Theme. Every differentiation problem corresponds to an antidifferentiation problem.

Differentiation Problems	Antidifferentiation Problems
$(x^2)' \quad = \quad 2x$	$\int 2x dx \quad = \quad x^2 + C$
$=$	$\int \cos(x) dx \quad =$
$(e^x + \ln(x))' \quad =$	$=$
$(\cos(x^2))' \quad =$	$=$
$(e^{\tan(x)})' \quad =$	$=$
$=$	$\int 4x \sin(x^2) dx \quad =$
$=$	$\int \cos(x) e^{\sin(x)} \quad =$

A *differential equation* is an equation that involves a function and its derivatives. An **initial value problem (IVP)** asks you to solve for a *particular* antiderivative based on a differential equation and an initial condition.

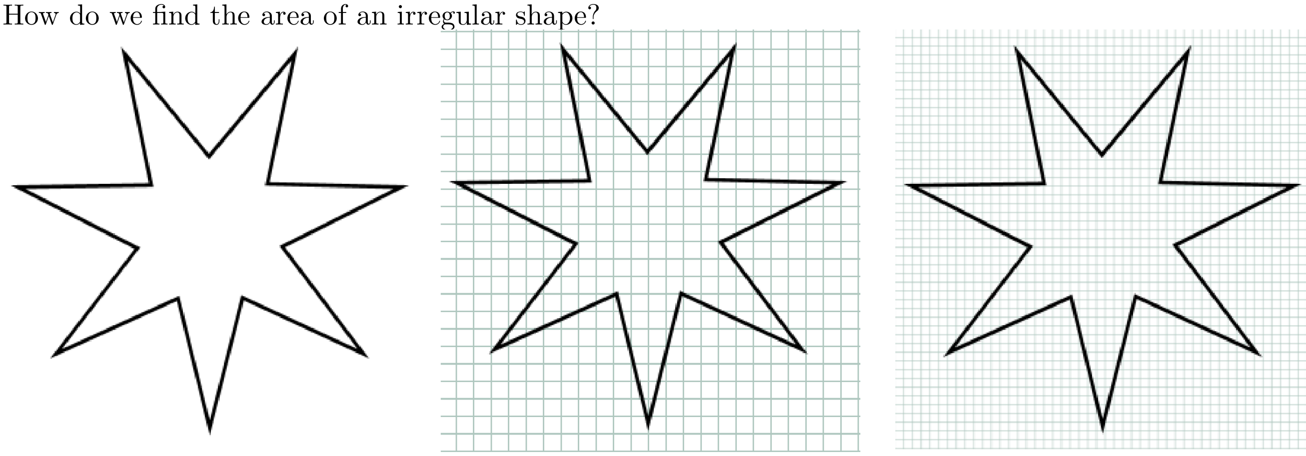
1. Find $s(t)$ if $\dfrac{ds}{dt} = \cos t + \sin t,\ s(\pi) = 1.$

2. Find $v(x)$ if $\dfrac{dv}{dx} = \dfrac{1}{2}\sec x \tan x,\ v(0) = 1.$

3. Find $y(t)$ if $\dfrac{d^2y}{dt^2} = \dfrac{3t}{8},\ \left.\dfrac{dy}{dt}\right|_{t=1} = 3$, and $y(4) = 4$.

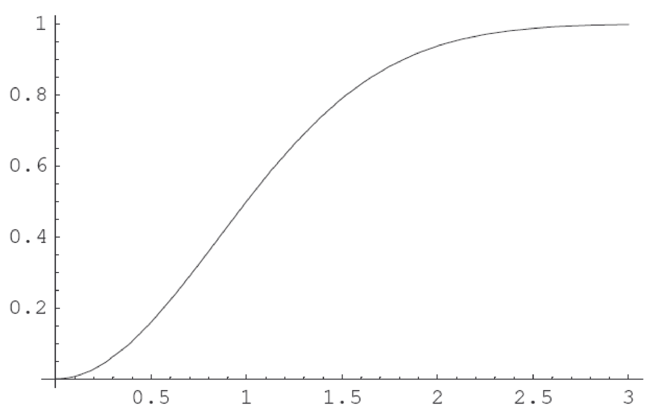
Subdivide–Approximate–Accumulate–Refine

How do we find the area of an irregular shape?



The same will be true for finding areas under curves, distance traveled, and average values of functions.

Area under a curve. Let's approximate the area under the curve $f(x) = 1 - 2^{-x^2}$ between $0 \le x \le 3$.



Number of subdivisions: $n =$
$\Delta x =$

lower sum:

upper sum:

midpoint rule:

To improve our approximation, increase the number of subintervals. (A programmable calculator comes in handy here. If you have a TI-83 or 84, I recommend http://math.ucsd.edu/~ashenk/Calculators/Riemann_TI-83.pdf.)

n	lower sum	upper sum	midpoint rule
10	1.78632...	2.08573...	1.93594...
50	1.90603...	1.96591...	1.93597...
100	1.92100...	1.95094...	1.93597...
250	1.93000...	1.94196...	1.93597...

Riemann Sum. For a function $f(x)$ on an interval $[a, b]$ the Riemann Sum

$$\sum_{k=0}^{n} f(x_i^*)\Delta x$$

approximates the (signed) area under the curve from $[a, b]$ using n intervals.

$\Delta x=$
Endpoints $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$:
Sample points x_i^* lies in ith subinterval $[x_{i-1}, x_i]$

The definite integral of f over $[a, b]$ is the limit of the Riemann sum $\sum_{k=0}^{n} f(x_i^*)\Delta x$ as $n \to \infty$ using *any* choice of x_i^* in $[x_{i-1}, x_i]$, provided the limit exists. If the limit exists, we say the function is *integrable on* $[a, b]$.

$$\int_a^b f(x)dx$$

A continuous function is always integrable, that is to say

Compute $\int_0^b c dx$ where b, c are fixed real numbers.

Compute $\int_0^b x dx$ where b is a fixed real number.

Properties of Definite Integrals

1. Order of Integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. Zero Width Interal: $\int_a^a f(x)dx =$

3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any number k

   $\int_a^b -f(x)dx = -\int_a^b f(x)dx$

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int$

6. Max-Min Inequality: If f attains a maximum and minimum value on the interval $[a, b]$ then

$$\min f \cdot (b-a) \leq \int_a^b f(x)dx \leq \max f \cdot (b-a)$$

7. Domination: If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Suppose that $\int_1^2 f(x)dx = -4$, $\int_1^5 f(x)dx = 6$, and $\int_1^5 g(x)dx = 8$. Find

$\int_2^5 f(x)dx$ $\int_1^5 [4f(x) - g(x)]dx$ $\int_2^2 f(x)dx$ $\int_5^1 [g(x) - f(x)]dx$

Calculus & Analytic Geometry I

The Fundamental Theorem of Calculus

Warm-up. What does definite integral $\int_a^b f(x)dx$ represent?

What if $f(x)$ is negative?

Compute $\int_2^4 (1-x)dx$

So a definite integral represents a *signed* area—

- where $f(x)$ is above the x-axis, the definite integral is the area
- where $f(x)$ is below the x-axis, the definite integral is the negative of the area.

Compute $\int_0^{2\pi} \sin(x)dx$.

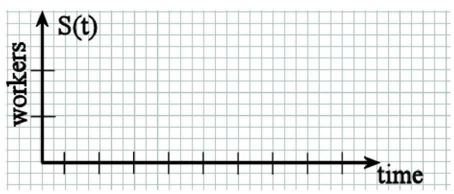
Another Acculmulation Problem. Four students are painting a house in shifts. The hours worked are shown below:

Worker	begin	end	hours worked
Chris	9 am	12 pm	
Toni	12 pm	4 pm	
Sam	10 am	2 pm	
Jo	2 pm	5 pm	

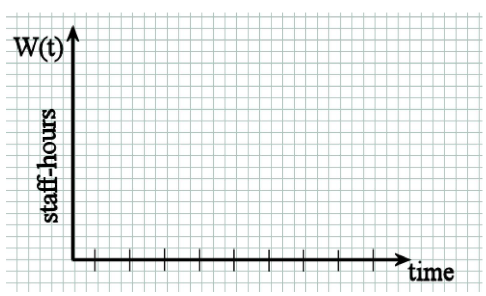
1. How many hours does each person work?

2. As the workers go through the day, they put in one token for each hour worked. How many tokens are there total at 5:00 pm? What does this represent?

3. Let $S(t)$ represent the number of people working at time t. Graph $S(t)$ verses time (9 to 5). What does the "area under this graph" represent?



4. Let $W(t)$ represent the work (or staff-hours) accumulated from 9 am until time t. Graph the function $W(t)$ versus time.



5. What is the relationship between the graph in part (3) and (4)?

Main Event. *Fundamental Theorem of Calculus, Part I.* If f is continuous on $[a,b]$, then its accumulation function $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b). Further more its derivative

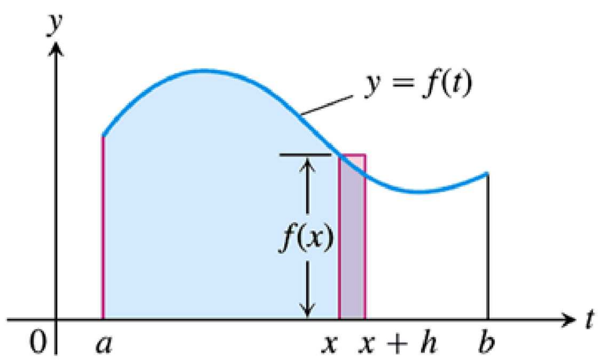
$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Illustrations. Find $\frac{dy}{dx}$

$y = \int_0^x \cos t dt$ $\qquad$ $y = \int_\pi^x \cos t dt$ $\qquad$ $y = \int_0^{e^x} \cos t dt$

Why? Let's interpret the difference quotient.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$



What is the difference between

$\int f(t)dt$ $\qquad$ and $\qquad$ $\int_a^x f(t)dt$?

Fundamental Theorem of Calculus, Part II. If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Illustrations.

$\int_0^\pi \cos t dt$ $\qquad$ $\int_0^{\ln 2} e^{3x}dx$ $\qquad$ $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}du$