Calculus & Analytic Geometry I

Limits Algebraically

Big Idea On *nice* functions, limits behave properly. (By that, I mean you can manipulate the symbol "lim" they way you would expect algebraically.)

The Laws. Suppose that c is a constant, n an integer, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

SUM $\lim\limits_{x\to a}[f(x)+g(x)] =$

DIFFERENCE $\lim\limits_{x\to a}[f(x)-g(x)] =$

CONSTANT MULTIPLE $\lim\limits_{x\to a}[cf(x)] =$

PRODUCT $\lim\limits_{x\to a}[f(x)g(x)] =$

QUOTIENT $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} =$

POWER $\lim\limits_{x\to a}[f(x)]^n =$

ROOT $\lim\limits_{x\to a}[f(x)]^{1/n} =$

$\lim\limits_{x\to a} c =$

$\lim\limits_{x\to a} x =$

Problem. Use the limit laws to evaluate:

$\lim\limits_{x\to -1} 3(x^2+1)^4$

$\lim\limits_{x\to 4^-} \sqrt{16-x^2}$

Other extremely useful properties.

- **Direct Substitution Property.** If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} =$$

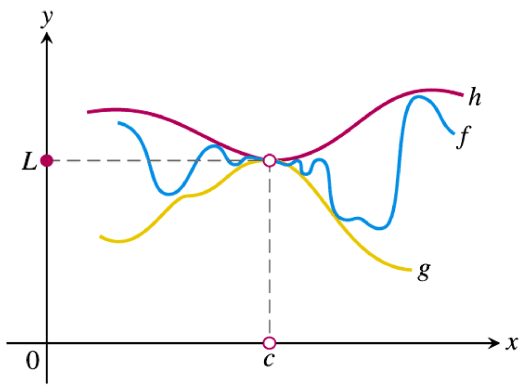
- If two functions are equal *except* at a fixed point a, their limits (if they exist) as they approach a must be equal.

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} \qquad\qquad \lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

- **The Sandwich Theorem.** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.



Example. Evaluate $\lim_{x \to 0} x^4 \cos \frac{2}{x}$

To think about over the break. It can be shown that the inequalities $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to 0. What does this tell you about $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$?

Calculus & Analytic Geometry I

Continuity

Nice functions are continuous....

Definition. A function $f(x)$ is *continuous at a(n interior) point* c, if

$$\lim_{x \to c} f(x) = f(c).$$

A function is *continuous at a left endpoint* a or is *continuous at a right endpoint* b if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b).$$

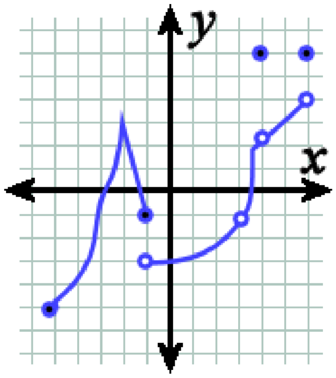
How can a function fail to be continuous at a point?

Definition. A function is called *continuous* if it is continuous at every point of its domain.

Is $f(x) = 1/x^2$ continuous?

Continuous functions include ... *Theorem 7, pg 124*

Where does the following function fail to be continuous on the interval $[-5, 6]$?



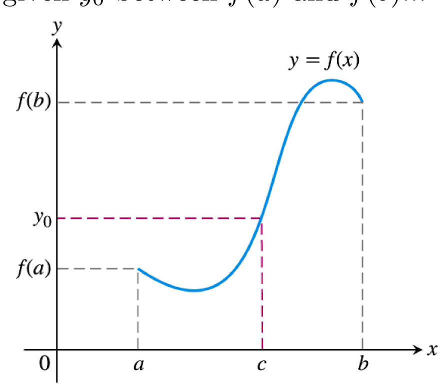
Good News—Algebraic combinations of continuous functions are continuous (whenever they are defined). So if f and g are continuous at $x = c$, then so are $f + g$, $f - g$, $f \cdot g$, $k \cdot f$ for a constant k, f/g provided $g(c) \neq 0$, and $f^{r/s}$ provided it is defined and r and s are integers. Inverses of continuous functions are continuous. Compositions of continuous functions are continuous.

Problems. For what real numbers do these functions fail to be continuous?

$$y = \frac{1}{x-2} - 3x \qquad\qquad y = \frac{1}{|x|+1} - \frac{x^2}{2}$$

$$y = \frac{\sin x}{x} \qquad\qquad y = \sqrt{2x+3}$$

Intermediate Value Theorem for Continuous Functions A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. Said another way, given y_0 between $f(a)$ and $f(b)$...



Problem. Prove that there is a root of the equation $x^4 + x = 3$ in the interval $(1, 2)$.