Calculus & Analytic Geometry I

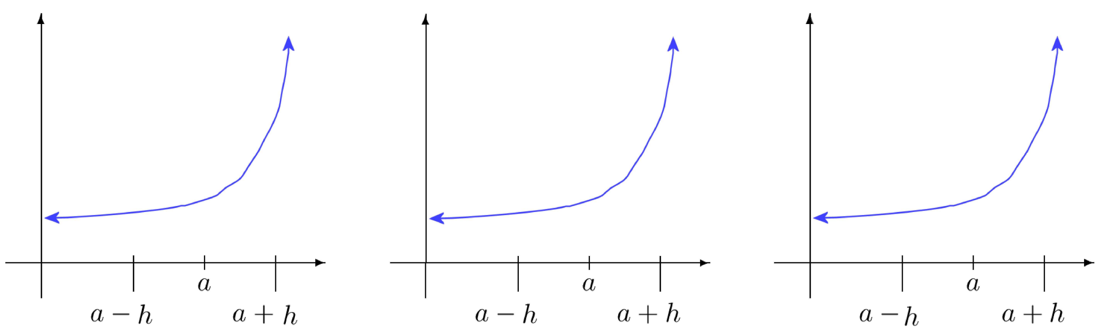
Tangents and Derivatives at a Point

Informally, the *derivative* of a function $y = f(x)$ at a point $x = a$ is defined to be the slope (or the rate of change) of the function at the point $(a, f(a))$. In §2.1, we estimated the rate of change (a.k.a. the derivative) by calculating the average rates of change of the function on smaller and smaller intervals containing a and watched the rates stabilize.

For each of the intervals given below, write an equation representing the average rate of change for the general function $y = f(x)$ on the specified interval. Then, on the graph provided sketch the line used to calculate the average rate of change on that interval.

Interval: $(a-h, a)$ $(a, a+h)$ $(a-h, a+h)$

Avg. rate
of change:



The expressions you have just discovered are referred to as difference quotients. *Each expression* acts as an estimate for the derivative $f'(a)$. To improve these estimates, let $h \to 0$. This leads us to the formal definition of the derivative.

Definition. The *derivative (slope, rate of change)* of a function f at $x = a$ is defined as:

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} \end{aligned}$$

provided these limits exist and are equal. In this case, f is said to be *differentiable* at $x = a$.

Are all three difference quotients really needed? Consider $f(x) = |x|$ at $x = 0$.

Problems. Find the slope of the function's graph at the given point. Then find the equation for the line tangent to the graph there.

1. $y = 2\sqrt{x}, \qquad (1, 2)$

2. $y = \dfrac{x-1}{x+1}, \qquad x = 0$

Summary. The following ideas are equivalent:

- the slope of $y = f(x)$ at $x = x_0$
- the slope of the tangent to the curve $y = f(x)$ at $x = x_0$
- the rate of change of $f(x)$ with respect to x at $x = x_0$
- the derivative $f'(x_0)$
- the limit of (any) difference quotient, $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

The Derivative as a Function

Problem. Find the slope of the function's graph at the given point. Then find the equation for the line tangent to the graph there.

1. $y = \dfrac{x-1}{x+1}, \qquad x = 0$

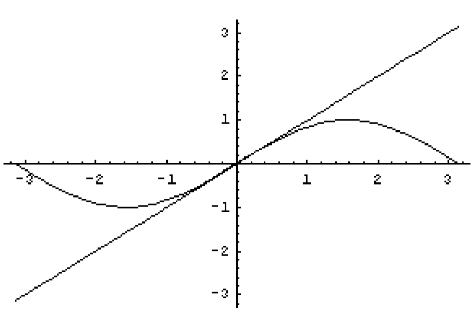
Working Backwards. What derivative is represented by each of the following expressions?

$$\lim_{x \to 5} \frac{2^x - 32}{x - 5} \qquad\qquad \lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$$

Derivative as a function. Up to now, we have looked at the derivative as a *number.* It gives us information about a function at a *point.* But the numerical value of derivative varies from point to point, and these new values can also be considered as the values of a new function— the derivative function—with its own graph. Viewed in this way the derivative is a global object.

All functions and their derivatives are related in the same way.

Function	$\leftrightarrow$	Derivative
increasing	$\leftrightarrow$	
decreasing	$\leftrightarrow$	
horizontal	$\leftrightarrow$	
steep (rising or falling)	$\leftrightarrow$	
gradual (rising or falling)	$\leftrightarrow$	
straight	$\leftrightarrow$	



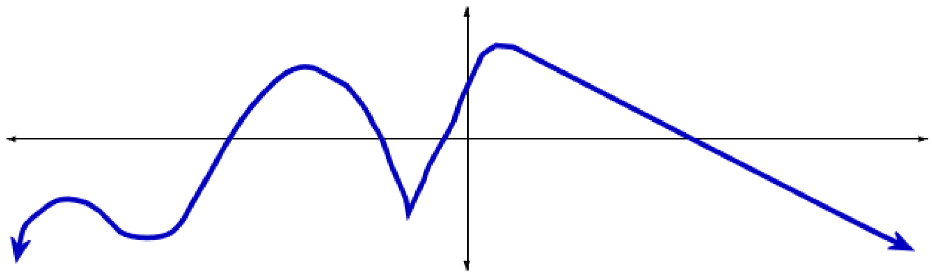
Notation. Many ways to denote the derivative of the function $y = f(x)$:

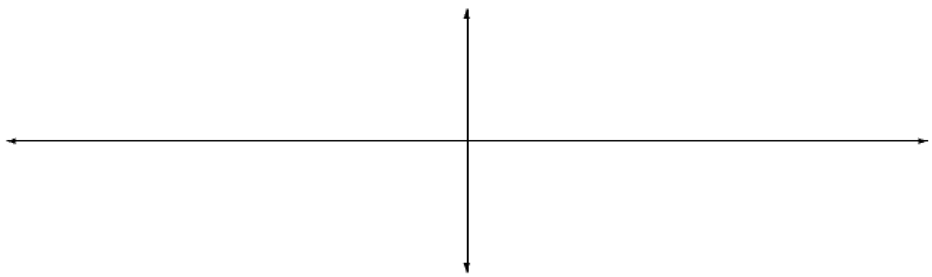
$$f'(y) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

To indicate the value of the derivative at a specified point $x = a$:

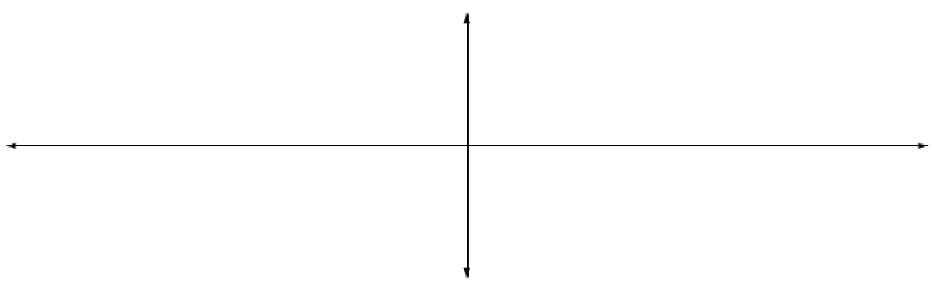
$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Illustration. For the graph below, sketch its derivative.





Now make a sketch of a graph which would have the original graph as its derivative.



As time permits: For each of the elementary functions below, first make a rough sketch of the function itself, then, based on the correspondences between functions and their derivatives, sketch the derivative of the function (as best you can.)

1. $f(x) = 3x$

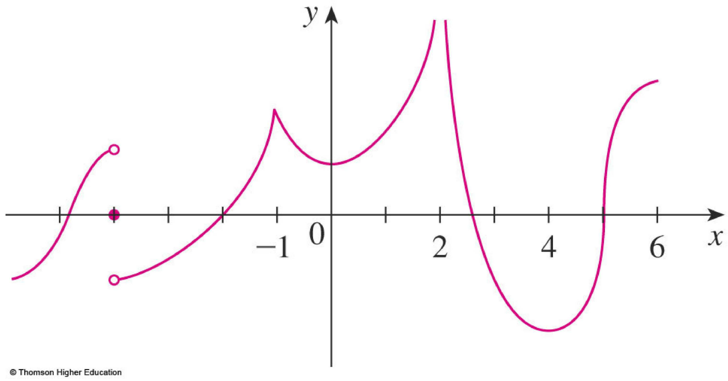
2. $g(x) = 3x + 1$

3. $h(x) = x^2$

4. $i(x) = e^x$

5. $j(x) = \sin x$

Definition. A function f is *differentiable at* a is $f'(a)$ exists. It is *differentiable on an open interval* (a, b) if it is differentiable at every number in the interval. A function is *differentiable* if it is differentiable at every point in its domain.



For your consideration

[T/F] If f is differentiable at a, then f is continuous at a.

[T/F] If f is continuous at a, the f is differentiable at a.

Higher Derivatives Since the derivative of a function f gives a new function f', there is nothing stopping us from analyzing the rate of change of f', denoted f'' or d^2f/dy^2. What about the rate of change of f''?