The Chain Rule—Derivative of function compositions

The Chain Rule — Motivation.

Suppose we are blowing up a spherical balloon. We know that the volume of a balloon depends its radius. ($V = \frac{4}{3}\pi r^3$.) When the radius is 15 cm, at what rate is the volume changing with respect to a change in the radius?

As we blow up the balloon, the radius (and hence the volume) are changing over time. Suppose the radius is changing at a rate of 3 cm every second when $r = 15$ cm. When the radius is 15 cm, at what rate is the volume changing with respect to a change in time?

The Chain Rule — Derivation.

The chain rule applies to a composition of functions. Suppose $f(g(x))$ is a composite function. Let us write

$$z = g(x) \text{ and } y = f(z), \text{ so } y = f(g(x)).$$

How does a change in x *approximately* effect a change in z?

How does a change in z *approximately* effect a change in y?

Combine these two approximations to relate a change in y to a change in x.

In words: The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

Problems. Find the derivatives of the given functions:

1. $f(x) = \sqrt{1 - x^2}$

2. $z = 3^{-6t}$

3. $h(r) = \sin(10r + 3)$

4. $p(x) = \sec x$

5. $\ell(\theta) = \sin 5\theta + \cos^2 \theta$

6. $g(s) = \sec^3(4s)e^{\sin s}$

7. The quotient rule as an application of the product rule and the chain rule.
   $P(x) = f(x) \cdot g^{-1}(x)$

Implicit Differentiation

Sometimes we are faced with equations that *imply* a relation between two variables.

$$x^2 + y^2 = 25 \qquad (x^2 + y^2)^2 = x^2 - y^2 \qquad x = \tan(y)$$

Sometimes we can solve for y in terms of x and sometimes we can't. But we can still ask about the rate of change of y with respect to x. We treat y as a function of x and use the chain rule.