Calculus & Analytic Geometry I

Linearization (A.K.A. Linear Approximation)

Linear Approximations. If a function f is "nice" at a point a, we can approximate the function near a by the equation of the line through $(a, f(a))$ with slope $f'(a)$. In other words, we use the equation of the tangent line to approximate the function near a.

> **Problem.** Make a rough sketch of the graph $f(x) = \sqrt{x}$. Find the linearization of $f(x)$ at the point $(4, 2)$.
>
> Estimate $\sqrt{4.0036}$.
>
> How close to the true value of $\sqrt{4.0036}$ is our estimate using the linear approximation?

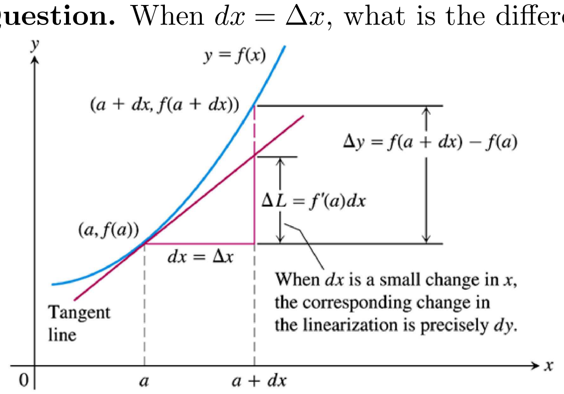
Differentials. Let $y = f(x)$ be a differentiable function. The **differential** dx is an *independent variable.* The **differential** dy is defined as

$$dy = f'(x)dx.$$

Problem

1. Find dy if $f(x) = \sqrt{x}$.
2. Find the value of dy when $x = 4$ and $dx = .0036$.

Question. When $dx = \Delta x$, what is the difference between Δy and dy?



So the linearization uses dy to approximate the true change Δy. Said another way...

$$f(a+\Delta x) = f(a) + \Delta y \approx f(a) + df.$$

Problems. Find dy for $y = (1+x)^k$ where k is a constant. Then estimate $(1.002)^{50}$ and $\sqrt[3]{1.009}$.

We can use differentials to analyze how error propagates through our calculations. Suppose we are calculating the area of a square by measuring the length of one of its sides and then using the formula $A = s^2$. Find dA when $s = 10$ cm.

If our measurement of the side is accurate to within 1 mm, how accurate is our calculation for the area?

The dA calculated above is the absolute error.
The *relative error* compares the absolute error to the calculated value.

The relative error in the measurement is $\dfrac{\Delta s}{s} = \dfrac{ds}{s} =$

The relative error in the area calculation is $\dfrac{dA}{A} =$

Local Approximation by Polynomials. For a function $f(x)$ the linear approximation of f near $x = a$ is the line

$$y = f(a) + f'(a)(x-a).$$

Find $y(a)$ and $y'(a)$.

A better approximation of $f(x)$ is a polynomial of higher order which exactly matches the values of the function and some more of its derivatives at the point $x = a$. These are called *Taylor polynomials.* The **nth-degree Taylor polynomial of f centered at a** is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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The Extremes

We are beginning Chapter 4: Applications of Differentiation. Today we examine the maximum and minimum values of functions (a.k.a. the extremes).

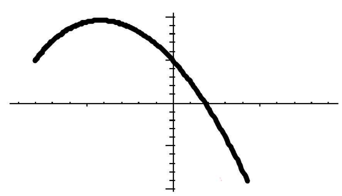
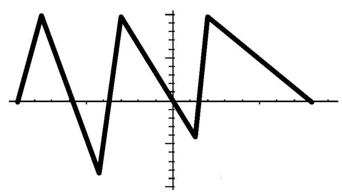
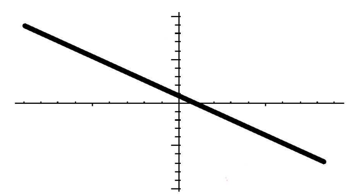
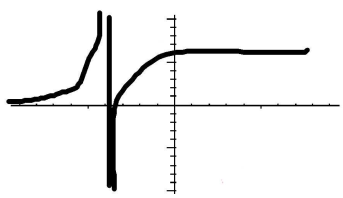
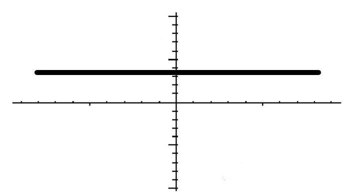
Definition. A function f with Domain D has an *absolute (global) maximum* at c if

$$f(x) \leq f(c) \text{ for all } x \text{ in } D$$

and an *absolute (global) minimum* at c if

$$f(x) \geq f(c) \text{ for all } x \text{ in } D.$$

The extreme value theorem says that a continuous function on a closed interval always has both an absolute maximum M and an absolute minimum m.

Definition. A function f with Domain D has a *local maximum* at c if

$$f(x) \leq f(c) \text{ for all } x \text{ in some open interval containing } c$$

and an *local minimum* at c if

$$f(x) \geq f(c) \text{ for all } x \text{ in an open interval containing } c.$$

What characteristics do extremes have?

Definition. A *critical point* is a point in the of the domain of a function f where either

- f' is zero, or
- f' is undefined.

Critical points are candidates for local extremes. Critical points and endpoints are candidates for global extremes.

Problems.

1. Find the absolute maximum and minimum value for $g(x) = xe^{-x}$ on the interval $-1 \leq x \leq 1$.

2. Find the absolute maximum and minimum value for $h(t) = 2 - |t|$ on the interval $-1 \leq t \leq 3$.