Calculus & Analytic Geometry I

Sigma notation and Definite Integrals

$$\sum_{k=1}^{n} a_k$$

Illustrations.

- $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$
- $\displaystyle\sum_{i=-1}^{4} 3 \cdot 2^{k+1}$
- $\displaystyle\sum_{j=0}^{2} \frac{(-1)^j}{j+1}$
- $\displaystyle\sum_{j=-1}^{1} \frac{(-1)^j}{j+2}$
- $\displaystyle\sum_{k=3}^{10} 1$
- $\displaystyle\sum_{i=1}^{10} n+i$

Rules for Finite Sums

1. Sum Rule: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.
2. Difference Rule: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$.
3. Constant Multiple Rule: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.
4. Constant Value Rule: $\sum_{k=1}^{n} c = n \cdot c$.

Some Important Sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Let $f(x) = 1 - x + 2x^2$. Find a formula for the upper sum obtained by dividing the interval $[1, 3]$ into n equal subintervals. Then take the limit of these sums as $n \to \infty$ to calculuate the area under curve on $[0, 3]$.

Riemann Sum. For a function $f(x)$ on an interval $[a,b]$ the Riemann Sum

$$\sum_{k=0}^{n} f(x_i^*)\Delta x$$

approximates the (signed) area under the curve from $[a,b]$ using n intervals.

$\Delta x=$
Endpoints $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$:
Sample points x_i^* lies in ith subinterval $[x_{i-1}, x_i]$

The definite integral of f over $[a,b]$ is the limit of the Riemann sum $\sum_{k=0}^{n} f(x_i^*)\Delta x$ as $n \to \infty$ using *any* choice of x_i^* in $[x_{i-1}, x_i]$, provided the limit exists. If the limit exists, we say the function is *integrable on* $[a,b]$.

$$\int_a^b f(x)dx$$

A continuous function is always integrable, that is to say

Compute $\int_0^b cdx$ where c is a fixed real number.

Compute $\int_0^b x^2dx$ where c is a fixed real number.

The Fundamental Theorem of Calculus

Warm-up. What does definite integral $\int_a^b f(x)dx$ represent?

Compute $\int_1^b x^2 dx$ by taking the limit of a lower Riemann Sum.

What if $f(x)$ is negative?

Compute $\int_2^4 (1-x)dx$

So a definite integral represents a *signed* area—

- where $f(x)$ is above the x-axis, the definite integral is the area
- where $f(x)$ is below the x-axis, the definite integral is the negative of the area.

Compute $\int_0^{2\pi} \sin(x)dx$.

Properties of Definite Integrals

1. Order of Integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. Zero Width Integral: $\int_a^a f(x)dx =$

3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any number k
   $$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int$

6. Max-Min Inequality: If f attains a maximum and minimum value on the interval $[a, b]$ then
   $$\min f \cdot (b-a) \leq \int_a^b f(x)dx \leq \max f \cdot (b-a)$$

7. Domination: If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Suppose that $\int_1^2 f(x)dx = -4$, $\int_1^5 f(x)dx = 6$, and $\int_1^5 g(x)dx = 8$. Find

$\int_2^5 f(x)dx$ $\int_1^5 [4f(x) - g(x)]dx$ $\int_2^2 f(x)dx$ $\int_5^1 [g(x) - f(x)]dx$

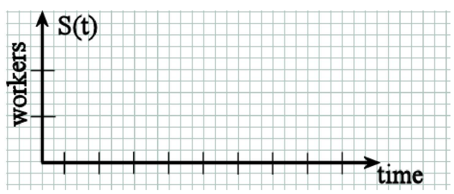
Another Acculmulation Problem. Four students are painting a house in shifts. The hours worked are shown below:

Worker	begin	end	hours worked
Chris	9 am	12 pm	
Toni	12 pm	4 pm	
Sam	10 am	2 pm	
Jo	2 pm	5 pm	

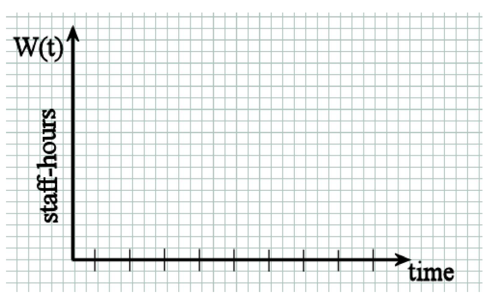
1. How many hours does each person work?

2. As the workers go through the day, they put in one token for each hour worked. How many tokens are there total at 5:00 pm? What does this represent?

3. Let $S(t)$ represent the number of people working at time t. Graph $S(t)$ verses time (9 to 5). What does the "area under this graph" represent?



4. Let $W(t)$ represent the work (or staff-hours) accumulated from 9 am until time t. Graph the function $W(t)$ versus time.



5. What is the relationship between the graph in part (3) and (4)?

Main Event. *Fundamental Theorem of Calculus, Part I.* If f is continuous on $[a,b]$, then its accumulation function $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b). Further more its derivative

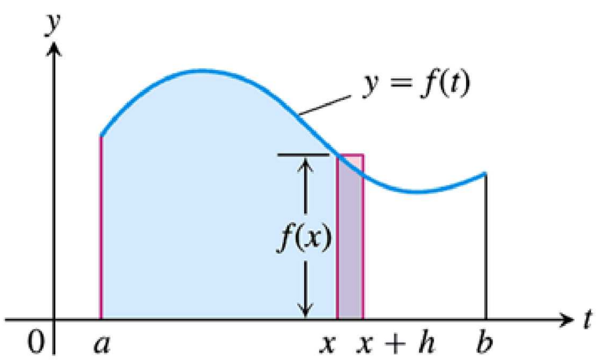
$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Illustrations. Find $\frac{dy}{dx}$

$y = \int_0^x \cos t dt$ $\qquad$ $y = \int_\pi^x \cos t dt$ $\qquad$ $y = \int_0^{e^x} \cos t dt$

Why? Let's interpret the difference quotient.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$



What is the difference between

$\int f(t)dt$ $\qquad$ and $\qquad$ $\int_a^x f(t)dt$?

Fundamental Theorem of Calculus, Part II. If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Illustrations.

$\int_0^\pi \cos t dt$ $\qquad$ $\int_0^{\ln 2} e^{3x} dx$ $\qquad$ $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}} du$