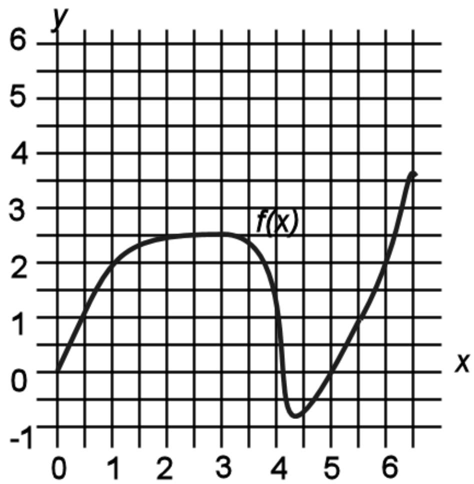
The Fundamental Theorem(s) of Calculus

Accumulation function. Suppose that $f(x)$ is a continuous function on $[a, b]$, let

$$g(x) = \int_a^x f(t)dt.$$

g is *accumulating* the signed area under f between a and x.



Example 1.

Use the graph above to compute the following:

$\int_1^3 f(t)dt$ $\int_1^5 f(t)dt$ $\int_1^1 f(t)dt$ $\int_1^0 f(t)dt$

The Fundamental Theorem of Calculus Part 1 (FTC1) explains the rate of change of an accumulation function.

Fundamental Theorem of Calculus I (FTC1). *If f is a continuous function on $[a, b]$, then the accumulation function g defined by*

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$.

Illustration. Let $g(x) = \int_1^x f(t)dt$ with f as in Example 1. Compute

$g'(3) =$ $\left.\frac{dg}{dx}\right|_{x=5} =$ $\left.\frac{d}{dx}\left(\int_1^x f(t)dt\right)\right|_{x=0} =$

Let's prove the FTC1 for a specific function.

$f(t) =$ Draw a rough sketch of $f(t)$.

Let $g(x) = \int_a^x f(t)dt =$

Find $g'(x)$ via the definition of derivative.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$$

Find the maximum and minimum values of f on $[x, x+h]$ and use the **Squeeze Theorem** to estimate

$$\leq \frac{1}{h} \int_x^{x+h} \underline{\hspace{4cm}} dt \leq$$

Conclusion. So $g'(x) =$

Question. How does our argument change if we move from a specific function like $f(t) = e^{-2t}$ to an arbitrary function $f(t)$? (*See page 382 of your book.*)

Practice Problems. Compute the following derivatives:

$$\frac{d}{dx} \int_5^x \frac{dt}{\ln t} \qquad \frac{d}{dz} \int_{-4}^z 27dt \qquad \frac{d}{dx} \int_1^{x^2} \tan(t)\csc(t)dt$$

Fundamental Theorem of Calculus II (FTC2). *If f is a continuous function on $[a, b]$ and F is any antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. This proof uses FTC1 and exploits the relationship between antiderivatives...

Let $g(x) = \int_a^x f(t)dt$. Then FTC1 implies that

$g'(x) =$

If F is any antiderivative of f, what is the relationship between F and g?

Now compute $F(b) - F(a)$.

Section 5.4 restates FTC2 as follows:

The Net Change Theorem. *The (definite) integral of a rate of change on an interval $[a, b]$ is the net change on the interval.*

Restating the Net Change Theorem more mathematically, let $F(x)$ be a continuous function on $[a, b]$...

Quick check. If $w(t)$ is the weight of a child in lbs, what does $\int_2^5 w'(t)dt$ represent?

More Practice Problems. Compute the following definite integrals:

$$\int_5^8 \frac{dt}{t} \qquad \int_{-4}^4 27dt \qquad \int_{\pi/4}^{\pi/3} \tan(t)\sec(t)dt$$

IMPORTANT! Know the distinction between

$$\int_a^b f(t)dt \qquad \int_a^x f(t)dt \qquad \int f(t)dt$$

Clearly, being able to calculate antiderivatives is going to be important. We already know:

$$\int k dx =$$

$$\int x^n = \begin{cases} & n \neq -1 \\ & n = -1 \end{cases}$$

$$\int e^x dx = \qquad \int a^x dx =$$

$$\int \sin x dx = \qquad \int \cos x dx =$$

$$\int \sec^2 x dx = \qquad \int \csc^2 x dx =$$

$$\int \sec x \tan x dx = \qquad \int \csc x \cot x dx =$$

$$\int \frac{1}{x^2+1} dx = \qquad \int \frac{dx}{\sqrt{1-x^2}} =$$

If you are facing an integral and you don't immediately know the antiderivative, rewrite, simplify, guess, and check.

1. $\int x^{-3}(x+1)dx$
2. $\int (1+\tan^2\theta)d\theta$
3. $\int \cos\theta(\tan\theta + \sec\theta)d\theta$
4. $\int \frac{\csc x}{\csc x - \sin x}dx$

1. $\frac{-1}{x} - \frac{1}{2x^2} + C$
2. $\tan\theta + C$
3. $-\cos\theta + \theta + C$
4. $\tan x + C$