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 CALCULUS & ANALYTIC GEOMETRY II
 

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## The Chain Rule and Substitution

**Warm-up.** Find the following derivatives

$$(e^{\sin x})' =$$

$$\frac{d}{dx} \ln |\sec x| =$$

$$(\cos(\ln x))' =$$

What integrals correspond to the problems above?

**Analogy**

*derivative:integral::chain rule: substitution*

**Recall** for any differentiable function  $u(x)$  and  $n \neq -1$ , the Chain Rule applied to the Power Rule says

$$\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \cdot \frac{du}{dx}.$$

So  $\int u^n \cdot \frac{du}{dx} dx =$

Let's try to apply this idea.

$$\int (2x + 3)^4 dx$$

$$\int \frac{ds}{\sqrt{5s-4}}$$

The general form of the Chain Rule says  $F(g(x))' = F'(g(x)) \cdot g'(x)$ . So

$$\int F'(g(x)) \cdot g'(x) dx =$$

**The Substitution Rule.** *If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

This allows us to replace a complicated integral by a (hopefully) simpler one. Let's see how this helps with our original problems of the day.

$$\int \cos x e^{\sin x} dx =$$

$$\int \tan x dx =$$

$$\int \frac{1}{x} \sin(\ln x) dx =$$

**Practice Problems.** Evaluate the following integrals.

1.  $\int \frac{4ydy}{\sqrt{2y^2 + 1}}$

2.  $\int \frac{6 \cos t}{(2 + \sin t)^3} dt$

3.  $\int \sec^2 x \tan x dx$

4.  $\int \sec^2 x \tan^2 x dx$

5.  $\int \sec^4 x \tan x dx$

6.  $\int \frac{\ln \sqrt{t}}{t} dt$

**The Substitution Rule for Definite Integrals.** If  $g'$  is continuous on  $[a, b]$  and  $f$  is continuous on the range of  $u = g(x)$ , then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

**Proof.** Let  $F$  be an antiderivative of  $f$ . So

$$\int_a^b f(g(x))g'(x)dx = F(g(x)) \Big|_a^b =$$

**Revisiting Practice Problems.** Evaluate the following definite integrals.

1.  $\int_0^2 \frac{4ydy}{\sqrt{2y^2+1}}$

2.  $\int_0^{\pi/2} \frac{6 \cos t}{(2 + \sin t)^3} dt$

3.  $\int_{\pi/6}^{\pi/4} \sec^2 x \tan x dx$