Calculus & Analytic Geometry II

More Applications of Integrals: Work and Average Values

Warm-up. Find the volume of a torus created by rotating a circle of radius 1 centered at $(2, 0)$ about the y-axis.

Ans: $4\pi^2$

Work. If a constant force F is applied to some object to move it a distance d, then the force has done *work* on the object provided the force applied *is parallel* to the motion. In this case the work done is given by

$$W = F \cdot d \qquad \textbf{(Constant-force formula for work)}.$$

Is work being done if:

we walk across the room holding a book?

we lift a book from the floor to a table?

Mass vs. Weight

Mass is the quantity of matter of an object—constant everywhere. **Weight** is the force exerted by gravity on an object—depends on the gravitational constant. Your mass is the same on Mercury, Mars, and Pluto but your weight would be very different.

Example How much work is done in lifting

(a) A 5-pound book 3 feet off the floor? (b) A 1.5 kilogram book 2 meters off the floor?

Ans: (a) 15 foot-pounds (b) 29.4 joules

If the force is variable or different parts of the object move different distances

-divide up the distance/object into small pieces

-assume the force/distance traveled is constant on each piece

-calculate the work for each piece and add (this is a Riemann Sum!)

-take the limit as the number of pieces grows without bound to get a definite integral.

Example A 28-meter uniform chain with a mass 2 kilograms per meter is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building?

Before you begin, how does this question differ from asking for the amount of work it would take to pull a 56 kilogram mass 28 meters to the top of the building.

Ans: 7683.2 joules

Example Calculate the work done in pumping oil to the rim of an inverted cone-shaped tank of height 20 m and base radius of 25 m. The depth of the oil in the tank is 10 m and oil has density 800kg/m^3.

Ans: 4.0×10^7 joules.

It is reported that the Great Pyramid of Egypt was built in 20 years. If the stone making up the pyramid has density 200 pounds per cubic foot, find the total amount of work done in building the pyramid. The pyramid is 410 feet high and has a square base 755 feet by 755 feet. (You might want to estimate how many workers were needed to build the pyramid to give the number some context. Assume a typical worker lifted ten 50 lb blocks 4 feet every hour.)

Ans: $1.6 \times 10^{1}2$ ft-lbs
13,000 workers!

In physics, **Hooke's law** of elasticity is an approximation that states that the amount by which a material body is deformed (the strain) is *linearly* related to the force causing the deformation (the stress). In particular, the force required to maintain a spring stretched x units beyond its natural length is

$$F(x) = kx$$

where k is a positive constant (called the spring constant).

Example Find the work done in stretching a spring by 0.1 m if $k = 8nt/m$.

0.04 *joules*

In general, is force is a function $F(x)$ of position x, then in moving from $x = a$ to $x = b$

$$\text{Work done} = \int_a^b$$

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Mean Value Theorem for Integrals and Average Value

Problem. What is the average of 98, 76, 95, and 89?

Suppose we want to know the average of a continuous function on an interval? Say what is the average value of the function $f(x) = 2x$ on $[1, 2]$?

Definition. The *average value of f* on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Problem. Find the average value of $h(x) = \cos^4 x \sin x$ on $[0, \pi]$.

Mean Value Theorem for Integrals. If f is continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx$$

that is, $\displaystyle\int_a^b f(x)dx = f(c)(b-a)$.

Problem. For $f(x) = \frac{2x}{(1+x^2)^2}$ find c such that $f(c)$ equals the average value of the function on $[0, 2]$.

Mean Value Theorem for Derivatives. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in the open interval so that the derivative at c equals the average rate of change on the interval $[a, b]$.

What's the relationship/connection between the two "Mean Value Theorems"? What does the Mean Value Theorem for derivatives say about accumulation functions $F(x) = \int_a^x f(t)dt$?