Calculus & Analytic Geometry II

Mean Value Theorem for Integrals and Average Value

Problem. What is the average of 98, 76, 95, and 89?

Suppose we want to know the average of a continuous function on an interval? Say what is the average value of the function $f(x) = 2x$ on $[1, 2]$?

Definition. The *average value of f* on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Problem. Find the average value of $h(x) = \cos^4 x \sin x$ on $[0, \pi]$.

Mean Value Theorem for Integrals. If f is continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

that is, $\displaystyle\int_a^b f(x)dx = f(c)(b-a)$.

Problem. For $f(x) = \frac{2x}{(1+x^2)^2}$ find c such that $f(c)$ equals the average value of the function on $[0, 2]$.

Mean Value Theorem for Derivatives. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in the open interval so that the derivative at c equals the average rate of change on the interval $[a, b]$.

What's the relationship/connection between the two "Mean Value Theorems"? What does the Mean Value Theorem for derivatives say about accumulation functions $F(x) = \int_a^x f(t)dt$?

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Techniques of Integration: Integration by Parts

Warm-up. Find $\frac{d}{dx}\left(x \sin x\right).$

Does this tell you anything about $\int x \cos x dx$?

For every differentiation technique there is a related antidifferentiation technique. Substitution is perhaps the most useful. *Integration by parts* will be the counterpoint to the product rule of derivatives.

chain rule: substitution :: product rule : integration by parts

Recall the product rule in all its glory:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Take the antiderivative of both sides to get:
$\int \frac{d}{dx}[f(x)g(x)] =$

Leading to the integration by parts formula:
$\int f(x)g'(x)dx =$

Many people prefer to make the substitution

$u = f(x)$

$v = g(x)$

Problem Reconsider $\int x \cos x dx$ through the lens of integration by parts. What are the possible assignments for u and dv? Which lead to *simpler* integrations?

$u =$ $dv =$ $u =$ $dv =$

$u =$ $dv =$ $u =$ $dv =$

Let's try a few more problems...

1. $\displaystyle\int (4x+5)e^{-x} dx$
2. $\displaystyle\int \frac{\ln(x)}{x^3} dx$
3. $\displaystyle\int \sin x \cos x dx$

Thought: Things can get ugly. Be aware of your surroundings at all times.

Multiple Applications: Although integration by parts should yield a simplified integral, it does not always simplify the problem *enough.* It may be necessary to use integration by parts several times.

Find $\displaystyle\int x^2 \sin(x)dx$.

Find $\displaystyle\int \cos(x)e^x dx$.

Find $\displaystyle\int \cos(\ln(x))dx$.

As time allows:

$\displaystyle\int \sin^{-1} x dx$ $\qquad$ $\displaystyle\int t \sec^2 2t dt$ $\qquad$ $\displaystyle\int t^3 e^{-t^2} dt$