What to do about troublesome integrals

Warm-up. Find an antiderivative of $\dfrac{e^x}{x}$.

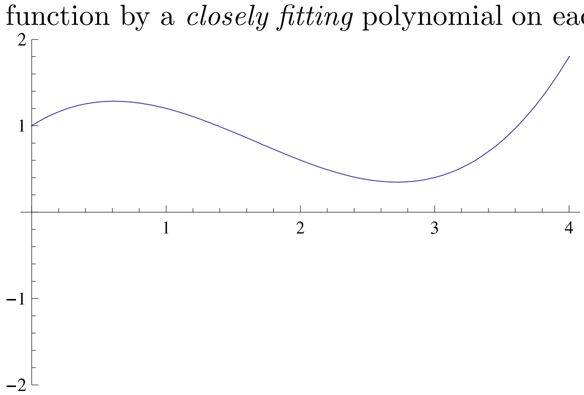
One possible answer is $F(x) = \displaystyle\int_0^x \frac{e^t}{t}dt$.

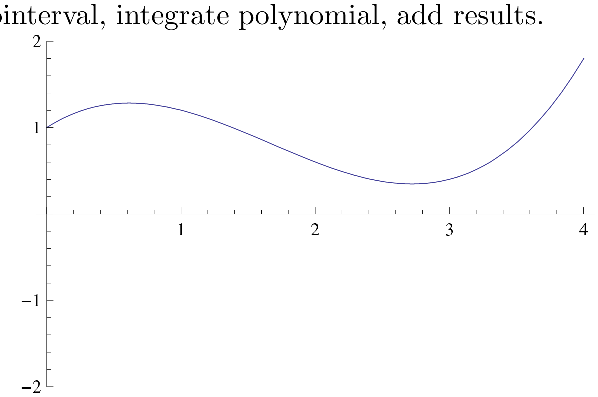
The majority of elementary functions don't have elementary antiderivatives. *Stewart, p. 488*

e^{x^2} $\sin(x^2)$ $\cos(e^x)$ $\dfrac{1}{\ln x}$

But we still might want to calculate definite integrals involving these functions. What should we do?

Big idea—Numerical Integration Partition the interval of integration, replace troublesome function by a *closely fitting* polynomial on each subinterval, integrate polynomial, add results.

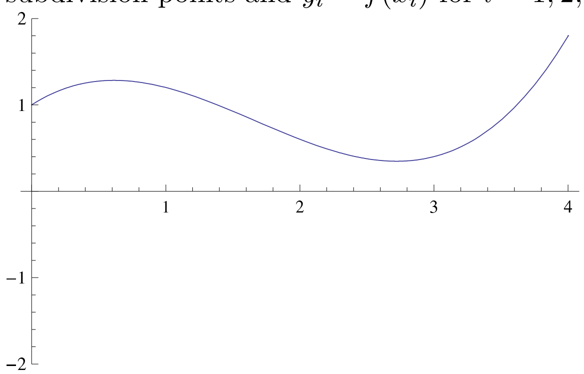




Trapezoidal Approximations

Find the area of a trapezoid with base Δx and heights y_{i-1} and y_i.

Let T be the approximation of $\int_a^b f(x)dx$ by n trapezoids where $a = x_0, x_1, x_2, \ldots, x_n = b$ are the subdivision points and $y_i = f(x_i)$ for $i = 1, 2, \ldots, n$. Then



$T =$

The Trapezoid Rule To approximate $\int_a^b f(x)dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y_is are the values of f at the partition points $x_0, x_1, \ldots x_n$ where $\Delta x = \frac{(b-a)}{n}$.

Example 1. Use the Trapezoid Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare this estimate with the exact solution.

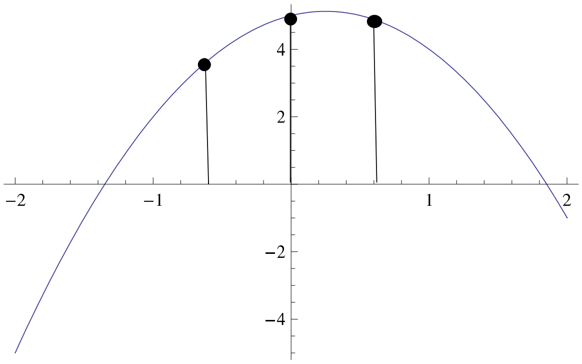
Ans: 75/32 , 7/3
% error: 0.00446

Calculus & Analytic Geometry II

Simpson's Rule: Approximations Using Parabolas

Warm-up. Any three points define a parabola. Find the parabola that passes through the points $(-1,4)$, $(0,1)$, and $(1,2)$.

Find the area under the parabola $y = Ax^2 + Bx + C$ passing through the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2). Express your answer in terms of y_0, y_1, and y_2.



How would the area change if the parabola were shifted horizontally?

So what is the area under the parabola $y = Ax^2+Bx+C$ passing through the points (x_2, y_2), (x_3, y_3), (x_4, y_4)?

Computing the area under all the parabolas and adding the results gives the approximation
$$\int_a^b f(x)dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2)+$$

Simpson's Rule. To approximate $\int_a^b f(x)dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y_is are the values of f at the partition points $x_0, x_1, \ldots x_n$ where $\Delta x = \frac{(b-a)}{n}$.

Example 2. Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$ and compare your result to the exact solution.

Ans: $32\frac{1}{12}$, *less than 3/10 %*

Error Estimates in the Trapezoidal and Simpson's Rules. If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, the the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \qquad \textit{Trapezoid}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of F from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \qquad \textit{Simpson's}$$

Example 2 continued. Approximate the error in example 2 using the above estimate.

Example 3. How large should n be to guarantee that Simpson's Rule approximation to $\int_0^1 e^{x^2} dx$ is accurate to within .00001?

$f^{(4)} = (12 + 48x^2 + 16x^4)e^{x^2}$

Ans: $n \geq 20$.