Calculus & Analytic Geometry II

Improper Integrals

Warm-up. Sketch the area represented by the following **improper integrals**

$$\int_1^\infty \frac{1}{x^3}dx \qquad\qquad \int_1^\infty \frac{1}{\sqrt{x}}dx$$

How can we make sense of an *infinite* area? How are these two integrals similar? How are they different?

To evaluate an integral when a limit of integration is infinite, fix a finite limit (say b), evaluate the definite integral, and then consider the limit of your solution as $b \to \infty$.

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}dx \qquad\qquad \int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{c\to\infty}\int_1^c \frac{1}{\sqrt{x}}dx$$

Definition of Improper Integrals of Type I. If $\int_a^t f(x)dx$ exists for every number $t \geq a$ then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

provided this limit exists as a finite number. In this case $\int_a^\infty f(x)dx$ is said to **converge.** Otherwise, we say that $\int_a^\infty f(x)dx$ **diverges**. We defined $\int_{-\infty}^b f(x)dx$ similarly.

Examples. Determine whether the following improper integrals converge or diverge.

$$\int_0^\infty e^{-5x}dx \qquad \int_{-\infty}^0 \frac{e^x}{1+e^x}dx \qquad \int_1^\infty \frac{1}{x^p}dx$$

In particular what values of p make this converge?

What happens if the limits of integration are $-\infty$ and ∞?

$$\int_{-\infty}^\infty x^3e^{-x^4}dx$$

Next Motivating Examples. Sketch the areas represented by the following integrals.

$$\int_0^2 \frac{1}{(x-2)^2}dx \qquad \int_{-1}^2 \frac{dx}{x^4}$$

Definition of Improper Integrals of Type II. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

provided this limit exists as a finite number.
If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

provided this limit exists as a finite number.
If f has a discontinuity at c where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Examples. Determine whether the following improper integrals converge or diverge.

$$\int_0^1 \ln x dx \qquad \int_0^2 \frac{1}{(x-2)^2}dx \qquad \int_{-1}^2 \frac{dx}{x^4}$$

Big idea! Sometimes is is difficult to find the exact value of an improper integral by antidifferentiation—but it may be possible to determine whether an integral converges or diverges. The key is to *compare* the given integral to one who behavior is already known.

Examples. Determine whether the following converge or diverge.

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx \qquad \int_4^\infty \frac{dt}{\ln t - 1}$$

Comparison Theorem. Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.
If $\int_a^\infty f(x)dx$ is ______________, then so is $\int_a^\infty g(x)dx$.
If $\int_a^\infty g(x)dx$ is ______________, then so is $\int_a^\infty f(x)dx$.

Investigate the convergence of $\int_1^\infty \frac{(\sin x)+3}{\sqrt{x}}dx$.

Useful Integrals for Comparison.

- $\int_1^\infty \frac{1}{x^p}dx$ converges for $p > 1$ and diverges for $p \leq 1$.
- $\int_0^1 \frac{1}{x^p}dx$ converges for $p < 1$ and diverges for $p \geq 1$.
- $\int_0^\infty e^{-ax}dx$ converges for $a > 0$.