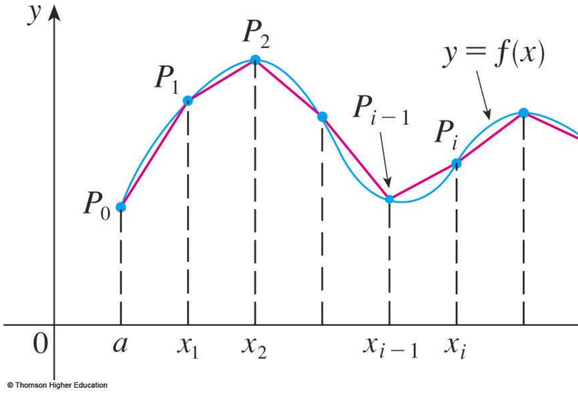
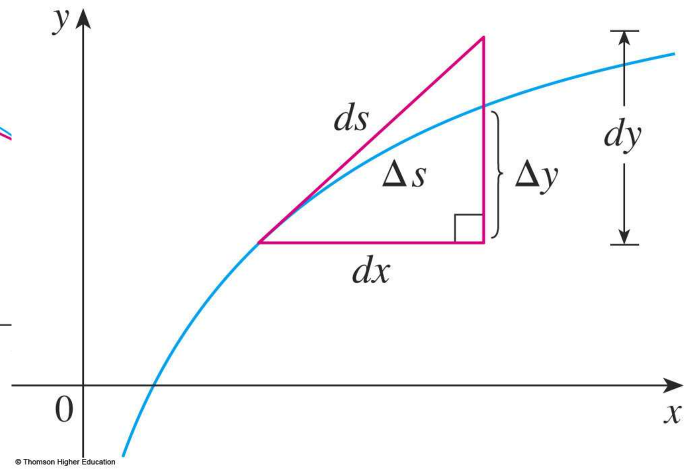
Calculus & Analytic Geometry II

Arc Length

Big Idea. To compute the length of the curve $y = f(x)$ from $x = a$ to $x = b$ (where $a < b$), divide the curve into small pieces, each one approximately a straight line.





$\Delta s \approx ds = \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx$

Arc length $\approx \sum$

For $a < b$, the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Arc length } = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

For $c < d$, the arc length of the curve $x = g(y)$ from $y = c$ to $y = d$ is given by

$$\text{Arc length } = \int_c^d \sqrt{1 + (g'(y))^2} dy.$$

Examples. Set up and evaluate integrals to compute the lengths of the indicated curves

1. $y = x^3$ from $x = 0$ to $x = 5$
2. $y^2 = 4(x + 4)^3$, $0 \le x \le 2$, $y > 0$.
3. $y = \ln(1 - x^2)$, $0 \le x \le 1/2$.

1)Numerical integration: 125.68
2) $\frac{2}{27}(55\sqrt{55} - 37\sqrt{37})$
3)Using partial fraction $-1 + \frac{1}{1+x} + \frac{1}{1-x}$ gives $\ln 3 - \frac{1}{2}$.

A new analogy

area: arc length::accumulation function: arc length function

Arc Length Function accumulates arc length from a fixed point.

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}dt.$$

Question. What is the rate of change of the arc length function $\frac{ds}{dx}$?

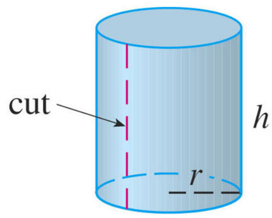
Problem. Find the arc length function for the curve $y = \sin^{-1} x + \sqrt{1 - x^2}$ starting with the starting point $(0, 1)$.

In general, integrals arising from arc length problems are *difficult*! Numerical techniques are often called for.

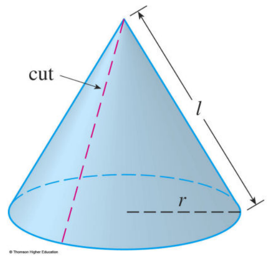
Calculus & Analytic Geometry II

Surfaces of Revolution

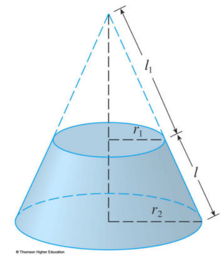
Everything Changes—Everything Stays the Same. To find the *volume* of a solid of revolution, we created a dV by rotating a small rectangular area, dA—creating either a disk, washer, or shell. Then we added them together in a Riemann sum, took the limit, and got a definite integral. To find the *surface area* of a solid of revolution, we create a dA by rotating a small piece of arc length ds—creating a slice of a cone. We then add them together in a Riemann sum, take a limit, and get a definite integral.



$2\pi rh$

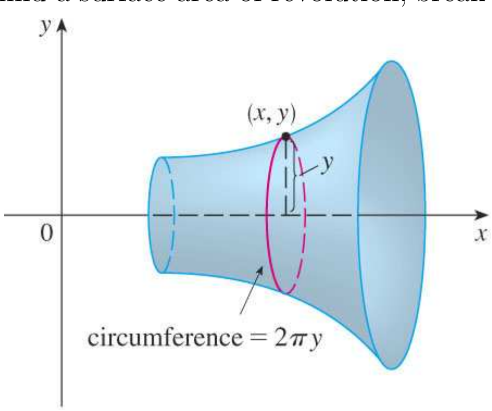


πrl

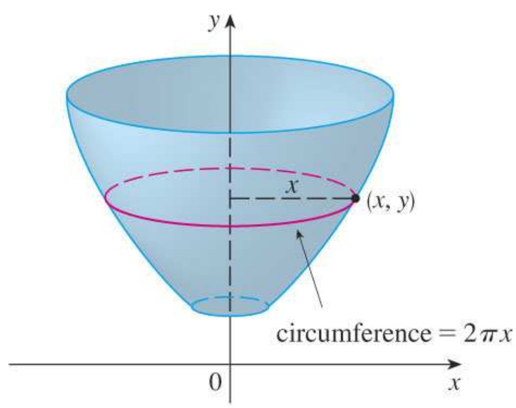


$2\pi\left(\frac{r_1+r_2}{2}\right)l$

So to find a surface area of revolution, break the surface into n "slices of cones" and add the areas.



(a) Rotation about x-axis: $S = \int 2\pi y\, ds$



(b) Rotation about y-axis: $S = \int 2\pi x\, ds$

Problems.

Rotate the curve $y = x^3$ from $0 \le x \le 2$ about the x-axis and find the resulting surface area.

Rotate the curve $y = 1 - x^2$ from $0 \le x \le 1$ about the y-axis and find the resulting surface area.