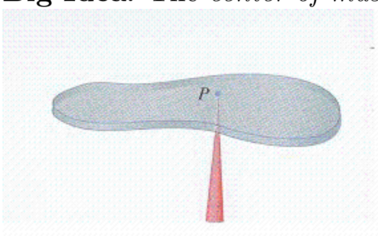
Calculus & Analytic Geometry II

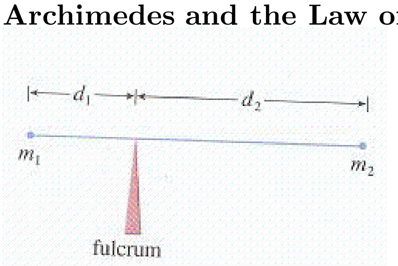
Moments and Center of Mass

Big Idea. The *center of mass* is the balance point.



Strategy. Break up the physical quantity into a large number of small parts, approximate each small part, add the results, take the limit, evaluate the resulting integral.

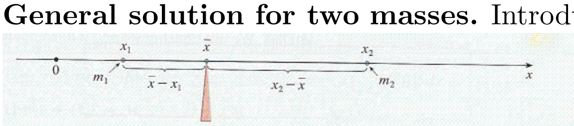
Archimedes and the Law of the Lever.



Masses m_1 and m_2 balance if $m_1 d_1 = m_2 d_2$.

Example. Find the balance point on a 12 foot board with a 20 lb kid on one end and an 80 lb kid on the other.

General solution for two masses. Introduce a coordinate system.



The center of mass will be $\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$
m_1x_1, m_2x_2 are called the *moments* of the masses m_1 and m_2 with respect to the origin.
$M = m_1x_1 + m_2x_2$ is called the *moment of the system*.
$m = m_1 + m_2$ is the total mass.

In general if we have n point masses m_i $(i = 1, 2, \ldots, n)$ located on the x axis at point x_i $(i = 1, 2, \ldots, n)$ then

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

Example. Find the moment and center of mass of the system with $m_1 = 6, m_2 = 5, m_3 = 10$ and $x_1 = 1$, $x_2 = 3$, $x_3 = -2$.

What happens when the masses are spread out in the plane? Example $m_1 = 6, m_2 = 5, m_3 = 10$ located at points $P_1(1, 5)$, $P_2(3, -2)$, $P_3(-2, -1)$ respectively.

$\bar{x}$ represents the line parallel to the y-axis that balances the weights.
$\bar{y}$ represents the line parallel to the x-axis that balances the weights.
The center of mass is $(\bar{x}, \bar{y})$.

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

numerator is M_y, moment of the system about the y-axis
measures tendency to rotate about the y-axis.

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

numerator is M_x, moment of the system about the x-axis
measures tendency to rotate about the x-axis.

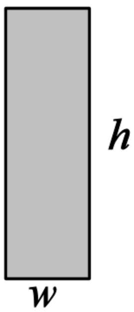
Calculus & Analytic Geometry II

Lamina

Definition. A *lamina* is thin plate, sheet, or layer.

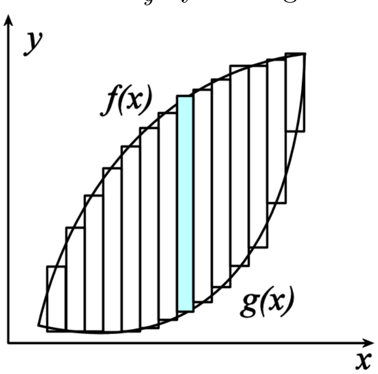
We will be considering lamina with *uniform density* ρ and seeking ways to determine its center of mass (sometimes called *centroid*.)

Consider a rectangle. What are the coordinates of its centroid?



This is the symmetry principle. If a region is symmetric across a line ℓ then the centroid lies on ℓ.

Suppose A is the region trapped between two curves $f(x)$ and $g(x)$. We can approximate the moment M_y by cutting the region into rectangular strips along the x-axis.



$$M_y \approx \sum_{i=1}^{n} \text{moment for } i\text{th slice} = \sum_{i=1}^{n} (\text{mass of } i\text{th slice})(\text{distance of centroid to } y\text{-axis})$$

$$M_x \approx \sum_{i=1}^{n} \text{moment for } i\text{th slice} = \sum_{i=1}^{n} (\text{mass of } i\text{th slice})(\text{distance of centroid to } x\text{-axis})$$

$$m = \int_a^b \rho(f(x) - g(x))dx \qquad \bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

Reiterating...

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$

When the density is constant over the lamina, it is irrelevant in the calculation! Only the area matters.

Examples. Find the center of mass for each of the given regions.

1. The region trapped between $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Ans: $(9/20, 9/20)$

2. The upper half of a circle of radius r.

Ans: $(0, \frac{4r}{3\pi})$