Calculus & Analytic Geometry II

Differential Equations

Big Idea. A mathematical *model* of a phenomenon is an abstract representation designed to capture certain features of interest. Calculus was developed to analyze models involving *rates of change.* These models are *differential equations*—involving a function and its derivatives. Using our knowledge of differential and integral calculus, we will sometimes be able to solve these models as initial value problems exactly. Other times we will have to settle for numerical approximations. Let's begin by examining what it means to be a solution to a differential equation.

Definition. A *first-order differential* equation is an equation

$$\frac{dy}{dx} = f(x, y)$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of first order because it involves only the first derivative $\frac{dy}{dx}$ (and not higher-order derivatives).

Example. Show that every member of the family of functions $y = C/x + 2$ is a solution of the first-order differential equation $\dfrac{dy}{dx} = \dfrac{1}{x}(2 - y)$ on the interval $(0, \infty)$, where C is any constant.

Example. Show that $y = (x + 1) - \dfrac{1}{3}e^x$ is a solution of the first-order initial value problem $\frac{dy}{dx} = y - x$, $y(0) = 2/3$.

Example. Show that $P(t) = \dfrac{1}{1 + 2e^{-kt}}$ is one solution to the differential equation $\dfrac{dP}{dt} = kP(1 - P)$.

Creating models can be fun!

Exponential Growth: The population grows at a rate proportional to its size.

Logistic Growth: Limited resources decrease the growth rate as the population approaches some carrying capacity, say K—though initially growth is (almost) proportional to population.

Newton's Law of Cooling/Heating: The change in temperature of an object is proportional to the difference in temperatures between the object and the ambient environment.

Coupled Populations: Lotka-Volterra Equations. Let $F(t)$ and $R(t)$ denote the number of foxes and rabbits at time t respectively and make the following assumptions:

In the absence of foxes, rabbits grow logistically.

The population of rabbits declines at a rate proportional to the product RF (death rate depends on the number of fatal encounters between the species and is approximately proportional to the both).

In the absence of rabbits the foxes die at a rate proportional to number of foxes present.

The population of foxes increases at a rate proportional to the number of encounters between rabbits and foxes.

Measles Epidemic. Let $S(t)$, $I(t)$, and $R(t)$ denote the number of people in a closed population that are susceptible to, infected by, or recovered from the measles at time t.

Susceptibles contract the disease at a rate proportional to both S and I. (Who is more likely to get infected, a person contacting 3 sick people every day for 2 days or 2 sick people every day for 3 days?)

Infecteds recover at a rate proportional to the number of infecteds. (If the disease runs its course for 14 days, and 28 people in the population are currently infected, how many do you expect to recover in the next day?)

The disease imparts permanent immunity and is not fatal!

Completing the Model. We are assuming that our total population is not changing. So

$$S'(t) + I'(t) + R'(t) =$$

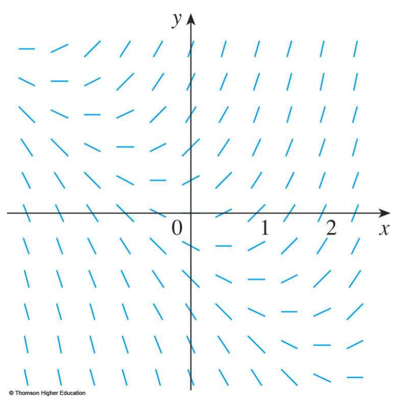
Every loss in I is due to a gain in R and every gain in I is due to a loss in S. So the complete SIR model is:

$$\begin{aligned} S'(t) &= -aS(t)I(t) \\ I'(t) &= aS(t)I(t) - bI(t) \\ R'(t) &= bI(t) \end{aligned}$$

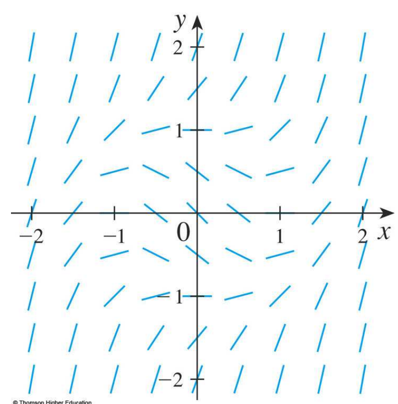
where a is the transmission coefficient and b is the recovery coefficient.

Bad News. It is impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. But we can learn a lot about the solution through a graphical approach.

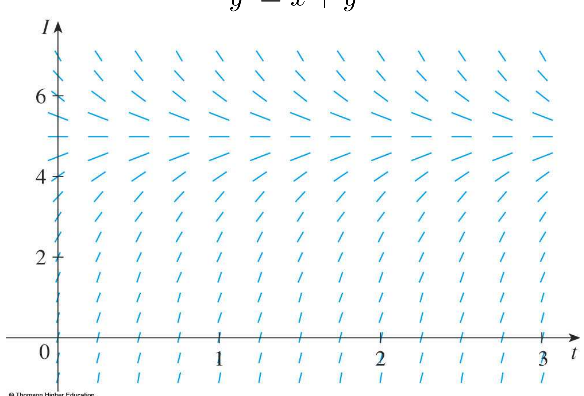
Slope Fields. Differential equations tell us the slope at any point (x, y). A direction field is a graph with short segments of slope $\frac{dy}{dx}$ at each point (x, y). (Visit `http://www.math.psu.edu/cao/DFD/Dir.html`.)



$y' = x + y$

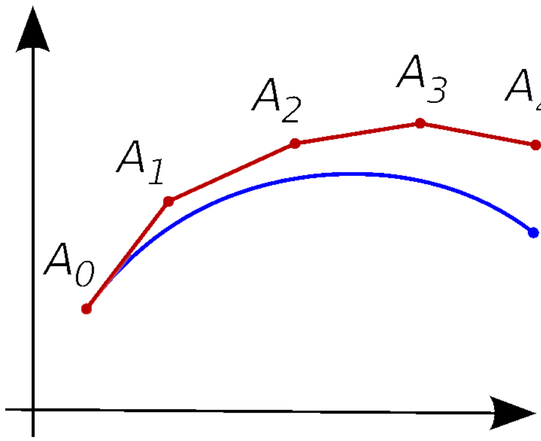


$y' = x^2 + y^2 - 1$



$y' = 15 - 3y$

Euler's Method. Use linear approximations (a.k.a. linearizations, tangent lines) to successively approximate output based on rate of change from given differential equations.



Example. Suppose $y' = 3y(1 - y/20)$ and $y(0) = 10$. Estimate $y(5)$ using $\Delta x = 2.5$.

x	0	2.5
$y(x)$	10	
$y'(x)$		
$dy = y'(x)dx$		

Recall: $y(a + dx) \approx y(a) + dy$

Question. How could we improve our estimates?

See applet `http://math.dartmouth.edu/~klbooksite/3.04/304.html`

Exploration. What will the population of susceptible, infecteds, and recovereds be for the model

$$\begin{aligned} S'(t) &= -.00001S(t)I(t) \\ I'(t) &= .00001S(t)I(t) - 1/14I(t) \\ R'(t) &= 1/14I(t) \end{aligned}$$

in a closed population of 50,000 with 2100 infected and 2500 recovered individuals? (Use a time step of 1 day.)

Estimates for the first three days using $\Delta t = 1$ day.

t	$S(t)$	$I(t)$	$R(t)$	$S'(t)$	$I'(t)$	$R'(t)$
0	454000.0	2100.0	2500.0	-953.4	803.4	150.0
1						
2						
3						

Question. How would the above process change, if we were to recalculate population sizes and rates of change after 1/2 a day rather than after an entire day?